Equazioni differenziali

Es. Trova $f(x)$ sapendo che   l'incognita è f!

$$x^2 + x f'(x) = 3 f(x)$$

Suggerimento: $f(x)$ è un polinomio di terzo grado.

$$f(x) = ax^3 + bx^2 + cx + d$$

$$x^2 + x\left(3ax^2 + 2bx + c\right) = 3\left(ax^3 + bx^2 + cx + d\right)$$

$$3ax^3 + (1+2b)x^2 + cx = 3ax^3 + 3bx^2 + 3cx + \underbrace{3d}_{=0}$$

$$\begin{cases} 1+2b = 3b \\ c = 3c \\ 0 = 3d \end{cases} \implies \begin{array}{l} b = 1 \\ c = 0 \\ d = 0 \end{array}$$

$$f(x) = ax^3 + x^2$$ ← SOLUZIONE GENERALE DI

$$x^2 + x f'(x) = 3 f(x)$$

$$x^2 + x\left(3ax^2 + 2x\right) = 3ax^3 + 3x^2 \quad ✓$$

ORDINE DELL'EQ. = massimo ordine di derivazione di f nell'eq.

= n. parametri della sol. generale.

Es. Trova $f(x)$ sapendo che

$$x^2 + x f'(x) = 3 f(x)$$

e $f(1) = 18$ ← condizione iniziale (*)

Sol.

$$f(x) = a x^3 + x^2$$

(*) $18 = a\, 1^3 + 1^2 \longrightarrow a = 17$

$$f(x) = 17 x^3 + x^2$$

Una soluzione particolare

Integrali indefiniti

Consideriamo l'equazione differenziale

$$f'(x) = \text{funzione data}$$

la soluzione di questa equazione si indica

$$f(x) = \int \text{funzione data} \; dx$$

e si chiama "integrale" o "primitiva" della funzione data.

Es. Risolvi l'equazione differenziale $f'(x) = 2x$

"calcola $\int 2x \, dx$"

$$f(x) = x^2 + C$$

$$(x^2 + C)' = \underbrace{(x^2)'}_{2x} + \underbrace{(C)'}_{0} = 2x$$

Integrali elementari

$$\int x^{\alpha}\, dx = \frac{1}{\alpha+1} x^{\alpha+1} + C \qquad \alpha \neq -1$$

$$\int \frac{1}{x}\, dx = \log_e(x) + C$$

$$\int \sin x\, dx = -\cos x + C \qquad \int e^x\, dx = e^x + C$$

$$\int \cos x\, dx = \sin x + C$$

FISSIAMO:

$$\int f(x)\, dx = F(x) + C$$

$$\int g(x)\, dx = G(x) + C$$

$$\int k \cdot f(x)\, dx = k \cdot F(x) + C$$

$$\int f(x) + g(x)\, dx = F(x) + G(x) + C$$

Eg. $\int (2x)^2 + e^x \, dx =$

$$= \int (2x)^2 \, dx + \int e^x \, dx$$

$$= \int 4x^2 \, dx + \int e^x \, dx$$

$$= 4\int x^2 \, dx + \int e^x \, dx$$

$$= 4\left(\frac{1}{3}x^3 + C_1\right) + \left(e^x + C_2\right)$$

$$= \frac{4}{3}x^3 + e^x \underbrace{+ 4C_1 + C_2}_{+C}$$

$$\int f(g(x)) \cdot g'(x) \, dx = F(g(x)) + C$$

Es. $\int x \cdot \sin(x^2)\, dx$

$$= \int \frac{1}{2} \cdot 2x \cdot \sin(x^2)\, dx$$

$F(g(x)) = -\cos(x^2)$

$$= \frac{1}{2} \int \underbrace{2x}_{g'(x)} \, \underbrace{\sin(x^2)}_{f(g(x))}\, dx$$

$$= \frac{1}{2} \cdot \left(-\cos(x^2)\right) + C = \frac{-\cos(x^2)}{2} + C$$

Suggerimento: calcolare la derivata per verifica!

Es $\int \sin(x) \cos(x) \, dx$

$$x \xmapsto{\; g \;} g(x) \xmapsto{\; f \;} f(g(x))$$

$$x \xmapsto{\; \sin x \;} \sin x \longmapsto \sin x$$

$$\int \underbrace{f(g(x))}_{\sin x} \cdot \underbrace{g'(x)}_{\cos x} \, dx = F(g(x)) + C$$

$$\frac{1}{2} (\sin x)^2 + C$$

$$\left(\frac{1}{2} (\sin x)^2 \right)' = \dots$$

$$\int \underbrace{\sin(x) \cos(x)}_{\frac{1}{2} \sin(2x)} \, dx = \int \frac{1}{2} \sin(2x) \, dx \quad (*)$$

$$x \xmapsto{\; g \;} 2x \xmapsto{\; f \;} \sin(2x)$$

$$\int \underbrace{f(g(x))}_{\sin(2x)} \cdot \underbrace{g'(x)}_{2} \, dx = -\cos(2x) + C$$

$$\int \frac{1}{2}\sin(2x)\,dx =$$

$$= \int \frac{1}{4}\cdot 2\sin(2x)\,dx = \frac{1}{4}\underline{\int 2\sin(2x)\,dx}$$

$$= -\frac{1}{4}\cos(2x) + C$$

$$\int \sin(x)\cos(x)\,dx =$$

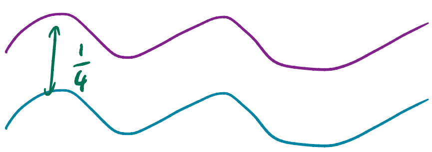
$\underline{\frac{1}{2}(\sin x)^2 + C}$ $\qquad\qquad$ $-\frac{1}{4}\cos(2x) + C$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =$

$(\cos x)^2 + (\sin x)^2 = 1$ $\qquad\qquad$ $-\frac{1}{4}\left(\underbrace{(\cos x)^2}_{1-(\sin x)^2} - (\sin x)^2\right) + C =$

$$= -\frac{1}{4}\left(1 - 2(\sin x)^2\right) + C$$

$$= \underline{\frac{1}{2}(\sin x)^2 - \frac{1}{4} + C}$$



$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

$$\int (2x)^2\,dx = \frac{1}{2}\cdot\frac{1}{3}(2x)^3 + C$$

Ex. $\displaystyle\int \frac{1}{2x+3}\,dx$

$$= \frac{1}{2}\log(2x+3) + C$$

Ex $\displaystyle\int 10^x\,dx =$ $\qquad 10 = e^{\log_e 10}$

$$= \int \left(e^{\log_e 10}\right)^x dx \qquad (a^b)^c = a^{b\cdot c}$$

$$= \int e^{(\log_e 10)\cdot x}\,dx$$

$$= \frac{1}{\log_e 10}\underbrace{e^{(\log_e 10)\cdot x}}_{10^x} + C = \frac{10^x}{\log_e 10} + C$$

$$\int b^x dx = \frac{1}{\log_e b} \cdot b^x + C$$

$$\int \frac{x-1}{x} dx = \int 1 - \frac{1}{x} dx =$$

$$= \int 1 dx - \int \frac{1}{x} dx$$

$$= x - \log x + C$$

$$\int 2\sqrt[3]{3x} dx = \int 2\sqrt[3]{3} \cdot x^{\frac{1}{3}} dx =$$

$$= 2\sqrt[3]{3} \int x^{\frac{1}{3}} dx$$

$$= 2\sqrt[3]{3} \frac{1}{1+\frac{1}{3}} \cdot x^{1+\frac{1}{3}} + C$$

$$= \frac{3\sqrt[3]{3}}{2} \sqrt[3]{x^4} + C$$

Es. $\int (2x)^2 + e^x \, dx$

Es. $\int x \cdot \sin(x^2) \, dx$

Es. $\int \sin(x) \cos(x) \, dx$

Es. $\int \frac{1}{2x+3} \, dx$

Es. $\int 10^x \, dx$