Nota Data una funzione classe $F : A \to B$ e un insieme $X \subseteq A$ esiste la funzione $f = F|_X : X \to F[X]$

$$a \mapsto F(a)$$

Teo (ricorsione transfinita)

Data una funzione classe $G : V \to V$ esiste un'unica* funzione classe $F : Ord \to V$ tale che

$$\forall \alpha \in Ord \quad F(\alpha) = G(F|_\alpha)$$

*ossia date F, F' come sopra, vale $\forall \alpha \in Ord \; F(\alpha) = F'(\alpha)$.

Prima di dimostrare il teorema, studiamone alcune conseguenze.

Ci serve la seguente osservazione

Oss. La funzione classe $Ord \to Ord$ succ : $\alpha \to \alpha + 1$ è crescente (quindi, in particolare, iniettiva).

Def Date classi A, B la classe $A \times B$ è

$$x \in A \times B \overset{def.}{\iff} \exists a \in A, b \in B \; x = (a, b)$$

$$x = \{\{a\}, \{a, b\}\}$$

Cor (ricorsione transfinita – v. 2)

Date funzioni classe $G_1 : Ord \times V \to V$

$G_2 : V \to V$

Data x_0

Esiste un'unica funzione $F : Ord \to V$ t.c.

$$F(0) = x_0$$

$$F(\alpha + 1) = G_1(\alpha, F(\alpha))$$

$$F(\lambda) = G_2(F|_\lambda) \quad \text{per } \lambda \text{ limite.}$$

Dim Provo che G non funziona.

$$G(f) = \begin{cases} \emptyset & \text{se } f \text{ non è una funz. o } Dom(f) \notin Ord. \\ x_0 & \text{se } f = \emptyset \\ G_1(\alpha, f(\alpha)) & \text{se } Dom(f) = \alpha + 1 \\ G_2(f) & \text{se } Dom(f) \text{ è limite} \end{cases}$$

$F(\emptyset) = G(F|_\emptyset) = G(\emptyset) = x_0$ ✓

$F(\alpha+1) = G(F|_{\alpha+1}) = G_1(\alpha, F|_{\alpha+1}(\alpha)) = G_1(\alpha, F(\alpha))$ ✓

$Dom(F|_{\alpha+1}) = \alpha+1 \Rightarrow$ III caso

$F(\lambda) = \dots$

Cor Def (operazioni sugli ordinali)

$F(x) = \alpha + x$
$\alpha + 0 = \alpha$ — $F(0) = \alpha$
$\alpha + (\beta+1) = (\alpha+\beta)+1$ — $F(\beta+1) = F(\beta)+1$
$\alpha + \lambda = \sup\{\alpha+\beta \mid \beta < \lambda\}$ — $F(\lambda) = \sup\{F(\beta) \mid \beta < \lambda\}$

$\alpha \cdot 0 = 0$	$\alpha^0 = 1$
$\alpha \cdot (\beta+1) = \alpha\cdot\beta + \alpha$	$\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$
$\alpha \cdot \lambda = \sup\{\alpha\cdot\beta \mid \beta < \lambda\}$	$\alpha^\lambda = \sup\{\alpha^\beta \mid \beta < \lambda\}$

Prop:

$(\alpha+\beta, <_{\alpha+\beta}) \simeq (\alpha, <_\alpha) + (\beta, <_\beta)$

idem per $\alpha\cdot\beta$, idem per α^β

(V.2)

Dim caso del +. Induzione transfinita su β.

- caso $\beta = 0$

$$\underbrace{\alpha + 0}_{\alpha} \sim \underbrace{(\alpha, <_\alpha) + (\emptyset, <_0)}_{(\alpha \sqcup \emptyset, <_+)}$$

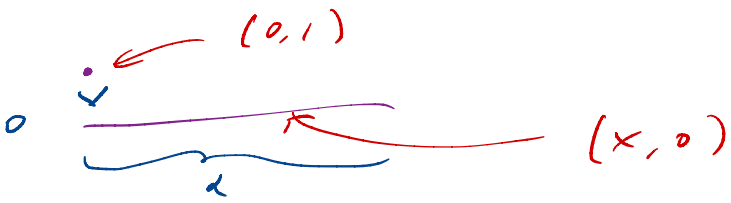
$$f : \alpha \to \alpha \times \{0\}$$
$$x \mapsto (x, 0)$$

$$\alpha \times \{0\} \cup \underbrace{\emptyset \times \{1\}}_{\emptyset} = \alpha \times \{0\}$$

- caso $\beta = \gamma + 1$

Oss $\quad \underbrace{\alpha + 1}_{\alpha \cup \{\alpha\}} \sim \underbrace{(\alpha, <_\alpha) + (1, <_1)}_{(\alpha \times \{0\} \cup \{(0,1)\}, <_+)}$

$$f : \alpha + 1 \to \sqcup 1$$
$$x \mapsto \begin{cases} (x,0) & \text{se } x \in \alpha \\ (0,1) & \text{se } x = \alpha \end{cases}$$

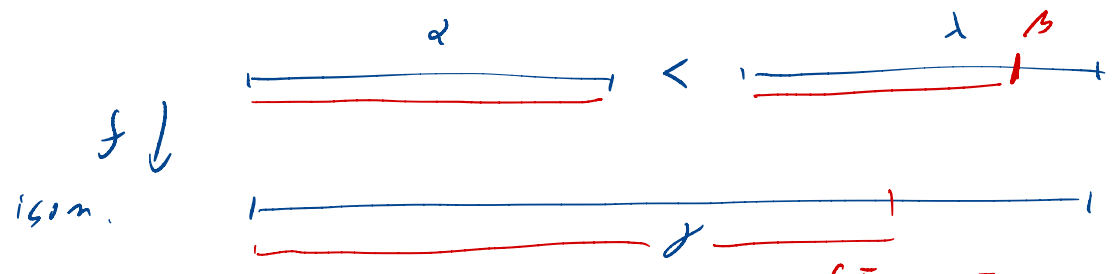


$$\underbrace{\alpha + (\gamma + 1)}_{(\alpha + \gamma) + 1} \sim \underbrace{(\alpha, <_\alpha) + (\gamma+1, <_{\gamma+1})}_{(\alpha,<) + ((\gamma,<) + (1,<))}$$

$$\sim \underbrace{\left((\alpha, <) + (\gamma, <)\right)}_{} + (1, <)$$

- $\alpha + \lambda \sim (\alpha, <_\alpha) + (\lambda, <_\lambda)$ con λ limite

Sia γ ordinale t.c. $\gamma \sim (\alpha, <_\alpha) + (\lambda, <_\lambda)$



$f[\alpha \cup \beta] = \alpha + \beta$

Perché?

$\alpha \cup \beta \sim \alpha + \beta$

$f[\alpha \cup \beta]$

$\alpha + \lambda = \sup \{ \alpha + \beta \mid \beta < \lambda \}$

$\lambda = \sup \lambda = \sup \{ \beta \mid \beta < \lambda \}$

$\alpha \cup \lambda = \sup \{ \alpha \cup \beta \mid \beta < \lambda \}$

$f[\alpha \cup \lambda] = \sup \{ f[\alpha \cup \beta] \mid \beta < \lambda \}$

$\gamma = \sup \{ \alpha + \beta \mid \beta < \lambda \} = \alpha + \lambda$

Esercizio: provare $\alpha \cdot \beta$ e α^β esercizio/noto ∎

Prop: Le funzioni classe

$(\alpha, \beta) \mapsto \alpha + \beta$ $(\alpha, \beta) \mapsto \alpha \cdot \beta$ $(\alpha, \beta) \mapsto \alpha^\beta$

sono

- crescenti nel secondo argomento (assumendo $\alpha \neq 0$ per $\alpha \cdot \beta$ e $1 < \alpha$ per α^β)
- mai decrescenti nel primo argomento.

Prop $\beta_1 < \beta_2 \to \alpha+\beta_1 < \alpha+\beta_2$

Per induzione su β_2

se $\beta_2 = 0$ vero a vuoto.

se $\beta_2 = \gamma+1$ $\qquad \beta_1 < \beta_2 = \gamma+1 \qquad \beta_1 \le \gamma$

$$\alpha+\beta_1 \le \alpha+\gamma < (\alpha+\gamma)+1 = \alpha+(\gamma+1) = \alpha+\beta_2$$

se β_2 è limite

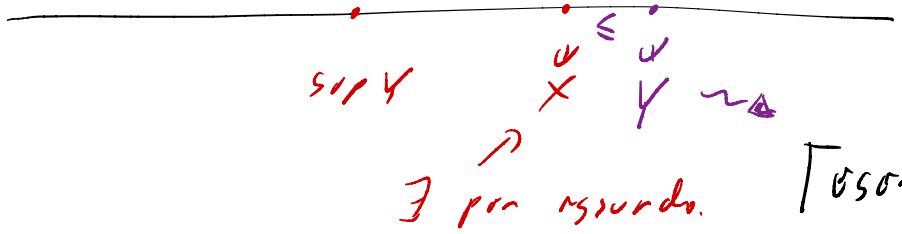
$$\alpha+\beta_1 < \alpha(\beta_1+1) \le \sup\{\alpha+\gamma \mid \gamma < \beta_2\} = \alpha+\beta_2$$

$\beta_1 < \beta_2 \to \beta_1+1 < \beta_2$

- $\alpha_1 < \alpha_2 \to \alpha_1+\beta \le \alpha_2+\beta$

Oss Siano X, Y insiemi non vuoti di ordinali

$$\forall \alpha \in X \ \exists \beta \in Y \ \alpha \le \beta \to \sup X \le \sup Y$$



$\exists$ per ognuno.

Esercizio: trovare cosa si guasta ragionando con $<$

caso $\beta = 0$ ovvio.

caso $\beta = \gamma+1$ $\qquad \alpha_1+\gamma \le \alpha_2+\gamma$ ip. induttiva

$$\alpha_1+\beta = (\alpha_1+\gamma)+1 \le (\alpha_2+\gamma)+1 = \alpha_2+\beta$$

caso β limite $\qquad \alpha_1+\gamma \le \alpha_2+\gamma$ ip. induttiva.

$$\alpha_1+\beta = \sup\{\alpha_1+\gamma \mid \gamma < \beta\} \le \sup\{\alpha_2+\gamma \mid \gamma < \beta\} = \alpha_2+\beta$$

∎

Teo (Forma normale di Cantor)

Ogni ordinale α si può scrivere come somma finita

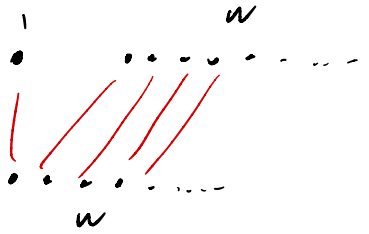
$$\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \ldots + \omega^{\beta_n} \cdot k_n$$

con $\beta_1 > \beta_2 > \ldots > \beta_n$ e $k_1 \ldots k_n \in \omega \setminus \{0\}$

Esercizio Scrivere in forma normale

- $1 + \omega = \omega^0 \cdot 1 + \omega^1 \cdot 1$ NON È FORMA NORMALE!

  $= \omega = \omega^1 \cdot 1$



$$\left.\begin{array}{l} \omega = 0 + \omega \leq 1 + \omega \leq \omega \\ 1 + \omega = \sup \{ 1 + n \mid n < \omega \} \leq \sup \omega = \omega \end{array}\right\} \Rightarrow 1 + \omega = \omega.$$

$1 + n \in \omega$

- $(\omega + 1) \cdot 2$

$$(n+1) \cdot 2 = (\omega + 1) \cdot (1 + 1) = (n+1) \cdot 1 + (n+1) \cdot 1$$

$$= (n+1) + (n+1) = \omega + (\omega + 1) + 1 = \omega + \omega + 1$$

$$= \omega \cdot 1 + \omega \cdot 1 + 1 = \omega \cdot (1 + 1) + 1 = \omega \cdot 2 + 1$$

- $(\omega+1)\cdot n$ con $n \in \omega \setminus \{0\}$

  $\overset{?}{=} \omega \cdot n + 1$

  - caso $n=1$ $\quad (\omega+1)\cdot 1 = \omega\cdot 1 + 1$ ✓
  - caso $n+1$

$$(\omega+1)\cdot(n+1) = \underline{(\omega+1)\cdot n} + (\omega+1) = \underline{\omega\cdot n + (1 + \omega)} + 1$$

$$= \omega\cdot n + \omega + 1 = \omega\cdot(n+1) + 1$$

- $(\omega+1)\cdot\omega = \sup\{(\omega+1)\cdot n \mid n<\omega\}$

$$= \sup\{\omega\cdot n + 1 \mid n<\omega\}$$

$\omega\cdot\omega$ = $\sup\{\omega\cdot n \mid n<\omega\} \leq$ $\qquad \leq \sup\{\omega\cdot(n+1) \mid n<\omega\}$ = $\omega\cdot\omega$

$$\omega\cdot n = \omega\cdot n + 0 < \omega\cdot n + 1 < \omega\cdot n + \omega = \omega\cdot(n+1)$$

$$\omega\cdot\omega \leqslant (\omega+1)\cdot\omega < \omega\cdot\omega$$

$$\Rightarrow (\omega+1)\cdot\omega = \omega\cdot\omega = \omega^2 = \omega^2\cdot 1$$

- $\underline{(\omega+1)^2} = (\omega+1)\cdot(\omega+1)$

$$= \underbrace{(\omega+1)\cdot\omega}_{(n+1)\cdot\omega} + (\omega+1) = \omega^2 + \omega + 1$$

$$= \omega^2\cdot 1 + \omega^1\cdot 1 + \omega^0\cdot 1$$

- $(w+1)^2 \cdot 2$
- $(w+1)^2 \cdot n$ con $n \in w \setminus \{0\}$
- $(w+1)^2 \cdot w$
- $(w+1)^3$
- $(w+1)^n$ con $n \in w$
- $(w+1)^w$

ALTRI ESERCIZI
(CIRCA IN ORDINE DI DIFFICOLTÀ)