Riassunto

- Esiste una f.c. $H : V \to Ord$
- α ord. (iniziale) $\longrightarrow H(\alpha)$ è il minimo ordinale iniziale β tale che $|\beta| \nleq |\alpha|$
- gli ordinali hanno $|\cdot|$ cardinalità $|\alpha| < |\beta|$

Oss. $\alpha, \beta \in Ord$ $\quad \beta < H(\alpha) \iff |\beta| \leq |\alpha|$

Dim. $\rightarrow (\beta, \in) \sim (X, <)$, $X \subseteq \alpha$ $\rightarrow |\beta| = |X| \leq |\alpha|$

$\leftarrow |\beta| \leq |\alpha| < |H(\alpha)| \rightarrow |\beta| < |H(\alpha)|$

$\downarrow$

$\beta < H(\alpha)$

$\beta \geq H(\alpha) \rightarrow \beta \supseteq H(\alpha) \rightarrow |\beta| \geq |H(\alpha)|$ ↯

Oss. Sia X insieme di ord. iniziali $\rightarrow \sup(X)$ è iniz.

Dim. Per assurdo $\beta < \alpha = \sup(X)$ con $|\beta| = |\alpha|$

$\beta < \sup(X) \Rightarrow \exists \gamma \in X \ \beta < \gamma$

$\beta \subseteq \gamma \subseteq \alpha \Rightarrow |\beta| \leq |\gamma| \leq |\alpha| = |\beta|$ ↯

$|\beta|$

Def $\quad w_0 = \omega$

$w_{\alpha+1} = H(w_\alpha)$

$w_\lambda = \sup \{ w_\alpha \mid \alpha < \lambda \}$

$w_0 = \omega$

$w_1 = H(w_0)$

$w_2 = H(w_1)$

$\vdots$

$w_\omega = \sup$ dei precedenti

$w_{\omega+1} = H(w_\omega)$

$\vdots$

Teo La f. classe $\alpha \mapsto w_\alpha$ è crescente e

$$\forall \beta \in Ord \ (\beta \text{ è iniziale}) \iff \exists \alpha \in Ord \ \beta = w_\alpha$$

Dim monotonia: esercizio

$\leftarrow$ per ind. transfinita

caso $\alpha+1$ $\quad w_{\alpha+1} = H(w_\alpha)$ e $H(\dots)$ è iniziale.

caso λ $\quad w_\lambda = \sup\{w_\alpha \mid \alpha < \lambda\}$ è iniziale per l'osservazione.

$\rightarrow$ Fissiamo β iniziale.

Esiste γ t.c. $\beta < w_\gamma$. Perché, per la monotonia,

$$\beta < \beta+1 \leq w_{\beta+1}$$

Sia γ minimo t.c. $\beta < w_\gamma$.

Osserviamo che γ è successore per continuità

(γ limite $\rightarrow$ $w_\gamma = \sup\{w_\delta \mid \delta < \gamma\} > \beta$

$\rightarrow \exists \delta < \gamma \ w_\delta > \beta \rightarrow$ minim. di γ)

Possiamo scrivere $\gamma = \alpha+1$

$$\beta < w_{\alpha+1} = H(w_\alpha) \Rightarrow |\beta| \leq |w_\alpha| \Rightarrow \beta \leq w_\alpha$$

per la min. di γ $\quad w_\alpha \leq \beta$ $\qquad \beta = w_\alpha$ $\blacksquare$

Notazione Usiamo il simbolo $\aleph_\alpha$ come abbreviazione per $|\omega_\alpha|$

Prop.
$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta) = \aleph_{\max(\alpha,\beta)}$$
$$\aleph_\alpha + |n| = \aleph_\alpha \cdot |n| = \aleph_\alpha \quad \text{per } n \in \omega \setminus \{0\}$$

Lemma $\forall \alpha \in Ord \quad \aleph_\alpha^2 = \aleph_\alpha$

Dim (proposizione — caso infinito)
stessa parola di prop. $\alpha \leq \beta$

$$\aleph_\beta \leq \aleph_\alpha + \aleph_\beta \leq \aleph_\beta \cdot 2 \leq \aleph_\beta \cdot \aleph_\alpha \leq \aleph_\beta^2 = \aleph_\beta$$

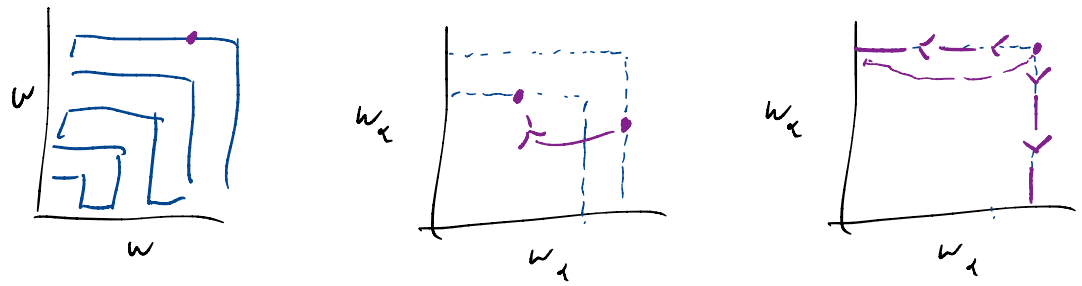
Oss. Un ord. inf. ω_α è limite.

Dim Assumiamo per assurdo $\omega_\alpha = \beta + 1$.
$\omega \leq \beta \rightarrow$ posso scrivere $\beta = \omega + \gamma$
ω_α
$$|\omega_\alpha| = |\omega + \gamma + 1| = |1 + \omega + \gamma| = |\omega + \gamma| = |\beta|$$
$$|\omega + \gamma + 1| = |1 + \omega + \gamma|$$

Oss. Se $\aleph_0 \leq |X| \leq \aleph_\alpha$ allora $\exists \beta < \alpha \quad |X| = \aleph_\beta$

Dim Esiste $f : X \rightarrow \omega_\alpha$ iniettiva.
$\Rightarrow X$ è ben ordinabile $a, b \in X \; a \leq b \overset{def.}{\hookrightarrow} f(a) < f(b)$
$\Rightarrow \exists \beta \quad |X| = \aleph_\beta$.

Dim (lemma)



$\omega_\alpha \times \omega_\alpha$ $\quad$ $\omega_\alpha \times \omega_\alpha$

$(x,y) \lhd (x',y') \overset{\text{def}}{\iff} \max(x,y) < \max(x',y')$

$\vee$

$\max(x,y) = \max(x',y') \wedge y < y'$

$\vee$

$\max(x,y) = \max(x',y') \wedge y = y' \wedge x < x'$

Asseriamo che $(\omega_\alpha \times \omega_\alpha, \lhd) \sim \omega_\alpha$

$\Rightarrow |\omega_\alpha \times \omega_\alpha| = |\omega_\alpha|$ $\quad$ ☺

nel b.o. prodotto $\omega_\alpha \times \omega_\alpha \times \omega_\alpha$

$(x,y) \lhd (x',y') \leftrightarrow (x, y, \max(x,y)) < (x', y', \max(x',y'))$

$\iota : \omega_\alpha \times \omega_\alpha \to \omega_\alpha \times \omega_\alpha \times \omega_\alpha$

$x, y \mapsto (x, y, \max(x,y))$

$a \lhd b \leftrightarrow \iota(a) < \iota(b)$

ord. di $\omega_\alpha \times \omega_\alpha \times \omega_\alpha$

Ci consegue $(\omega_\alpha \times \omega_\alpha, \lhd) \sim (\iota[\omega_\alpha \times \omega_\alpha], <)$.

Quindi è un b.o., quindi $\exists \beta \in \mathrm{Ord}$ $\beta \simeq (\omega_\alpha \times \omega_\alpha, \lhd)$

$|\omega_\alpha| \leq |\omega_\alpha \times \omega_\alpha| = |\beta| \Rightarrow \omega_\alpha \leq \beta$

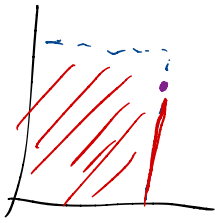
Dobbiamo escludere $\omega_\alpha < \beta$

Se $\omega_\alpha < \Lambda$ allora $\omega_\alpha \sim (\omega_\alpha \times \omega_\alpha, \prec)_{(x,y)}$

$(\omega_\alpha \times \omega_\alpha, \prec)_{(x,y)} \subseteq \max(x,y) \times \max(x,y)$

$\mu \in \omega_\alpha \to |\mu| < |\omega_\alpha|$

$|\mu| \times |\mu| = |\mu|$



$|(\omega_\alpha \times \omega_\alpha, \prec)_{(x,y)}| \leq |\mu| < |\omega_\alpha|$ ↯ ☐

Esercizi:

- Per $n \in \omega$ sia $\mathcal{P}^n(X) \overset{\text{def.}}{=} \{Y \in \mathcal{P}(X) \mid |Y| = n\}$
  $\forall n \in \omega \quad |\mathcal{P}^n(\omega_\alpha)| = \aleph_\alpha$
- $|\bigcup \{\mathcal{P}^n(\omega_\alpha) \mid n \in \omega\}| = \aleph_\alpha$
  Suggerimento: $|\text{sottoins. finiti di } \omega_\alpha| \leq |\{f : n \to \alpha \mid n \in \omega\}|$
  fissa $g : \omega_\alpha \times (\omega_\alpha \cup \{\clubsuit\}) \to \omega_\alpha \cup \{\clubsuit\}$ iterativa. Definisci $h : \bigcup \to \omega_\alpha$ con $h(\emptyset) = \clubsuit$, $h(f) = g(f(0), h(x \mapsto f(x+1)))$.
- $\alpha, \beta \in \text{Ord} \quad |\alpha| = |\beta| = \aleph_\gamma \longrightarrow |\alpha + \beta| = |\alpha \cdot \beta| = |\alpha^\beta| = \aleph_\gamma$.
- Se $|X| = \aleph_\alpha$ e $f : X \to Y$ è una funzione, $|f[X]| \leq \aleph_\alpha$.
- Se $|X \cup Y| = \aleph_\alpha$ e $|Y| < \aleph_\alpha$ allora $|X| = \aleph_\alpha$.
- $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$
  Suggerimento: $|\omega_{\alpha+1}| < |\mathcal{P}(\omega_{\alpha+1})| \leq |\mathcal{P}(\mathcal{P}(\omega_\alpha \times \omega_\alpha))|$