Esempio $\aleph_\omega < \aleph_\omega^{\aleph_0}$

$$\aleph_\omega = \sum_{\alpha<\omega} \aleph_\alpha < \prod_{\alpha<\omega} \aleph_{\alpha+1} \leq \prod_{\alpha<\omega} \aleph_\omega = \aleph_\omega^{\aleph_0}$$

$$\aleph_\omega = \max\left(\aleph_0, \sup_{\alpha\in\omega} \omega_\alpha\right) = \aleph_\omega$$

Esercizio (facile) Se $2^{\aleph_0} = \aleph_{41}$ allora $\aleph_{41}^{\aleph_0} = \aleph_{41}$

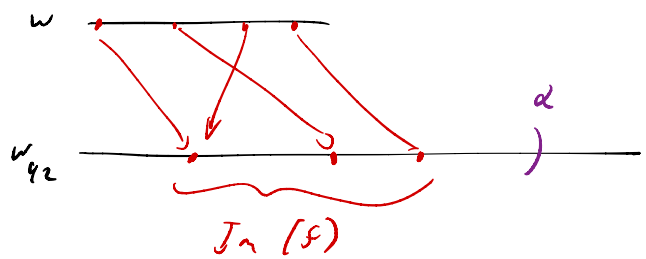
$$\aleph_{41}^{\aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \aleph_{41}$$

Esercizio (difficile) Se $2^{\aleph_0} = \aleph_{41}$ allora $\aleph_{42}^{\aleph_0} = \aleph_{42}$

$\aleph_{42} \leq \aleph_{42}^{\aleph_0}$ occorre dimostrare $\geq$

$\aleph_{42}^{\aleph_0} = |S|$ con $S = \{f : \omega \to \omega_{42}\}$

Fissiamo una $f : \omega \to \omega_{42}$



Idea: esiste $\alpha < \omega_{42}$ (che dipende da f) t.c. $\forall x \in \omega \;\; f(x) < \alpha$.

$$\sup\left(\mathrm{Im}(f)\right) = \bigcup \mathrm{Im}(f)$$

$$\left|\bigcup \mathrm{Im}(f)\right| = \left|\bigcup \{f(i) \mid i \in \omega\}\right| \leq \sum_{i\in\omega} \overbrace{|f(i)|}^{\leq \aleph_{41} \;\Leftrightarrow\; < \aleph_{42}}$$

$$\leq$$

$$\aleph_{41} = \aleph_0 \cdot \aleph_{41} = \sum_{i\in\omega} \aleph_{41}$$

Abbiamo dimostrato che

$$\forall f : \omega \to \omega_{\omega_2} \quad \exists \alpha < \omega_{\omega_2} \quad \forall x \in \omega \quad f(x) < \alpha$$

Ossia

$$\{ f : \omega \to \omega_{\omega_2} \} = \bigcup_{\alpha \in \omega_{\omega_2}} \{ f : \omega \to \alpha \}$$

$$\aleph_{\omega_2}^{\aleph_0} = | \quad | \leq \sum_{\alpha \in \omega_{\omega_2}} | \quad | = \sum_{\alpha \in \omega_{\omega_2}} |\alpha|^{\aleph_0}$$

$$\aleph_{\omega_2} = \aleph_{\omega_2} \cdot \aleph_{\omega_1} = \sum_{\alpha \in \omega_{\omega_2}} \aleph_{\omega_1} = \sum_{\alpha \in \omega_{\omega_2}} \aleph_{\omega_1}^{\aleph_0}$$

Esercizio $2^{\aleph_0} = \aleph_1 \to \forall n \in \omega \quad \aleph_{n+1}^{\aleph_0} = \aleph_{n+1}$

Se $2^{\aleph_0} = \aleph_1$:

$$\aleph_0^{\aleph_0} = \aleph_1 > \aleph_0$$

$$\aleph_1^{\aleph_0} = \aleph_1$$

$$\vdots$$

$$\aleph_n^{\aleph_0} = \aleph_n$$

$$\vdots$$

$$\aleph_\omega^{\aleph_0} > \aleph_\omega$$

Def (cofinalità - v.1)

Sia k cardinale infinito. $\mathrm{cof}(k)$ = il minimo cardinale λ t.c. esiste $\{k_i\}_{i<\lambda}$ con $\forall i<\lambda \ k_i < k$ e

$$\sum_{i<\lambda} k_i = k.$$

Esempi:

$\mathrm{cof}(\aleph_0) = \aleph_0$

$\mathrm{cof}(\aleph_\omega) = \aleph_0$ $\quad \aleph_\omega = \sum_{i<\omega} \aleph_i$

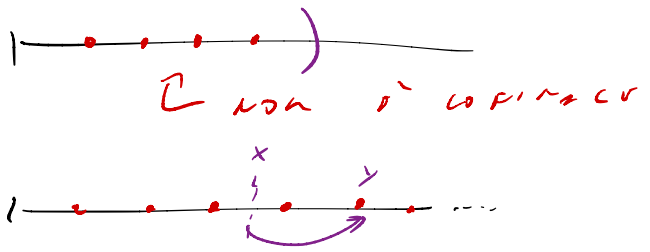
$\mathrm{cof}(\aleph_{\omega_2}) = \aleph_{\omega_2}$ $\quad \aleph_{\omega_2} = \sum_{i<\lambda} k_i \leq \sum_{i<\lambda} \aleph_{\omega_1} = |\lambda| \cdot \aleph_{\omega_1}$

$\aleph_{\omega_2}$ ↓

Def (cofinalità - v.2)

Dato $(S, <)$ totalmente ordinato, diciamo che $A \subseteq S$ è cofinale in S se $\forall x \in S \ \exists y \in A \ x \leq y$.

$\mathrm{cof}(S, <)$ è la minima cardinalità di un insieme A cofinale in $(S, <)$.



Esempi:

- $\mathrm{cof}(\omega) = \aleph_0$ $\quad A \subseteq \omega$ finito $\to$ posto $x = 1 + \max A$ e $\nexists y \in A \ x \leq y$.

Oss S è sempre cofinale in se stesso.

- $\mathrm{cof}(\mathbb{R}, \leq_{\mathbb{R}}) = \aleph_0$ — $\mathbb{N}$ è cofinale in $\mathbb{R}$

$\forall x \in \mathbb{R} \quad \exists y \in \mathbb{N} \quad x \leq y$

- $\mathrm{cof}([0,1], \leq_{\mathbb{R}}) = 1$ — $\{1\}$ è cofinale in $[0,1]$
- $\mathrm{cof}(n+1) = 1$ — $\{n\}$ ——————
- $\mathrm{cof}(\omega_\omega) = \aleph_0$ — $\{\omega_0, \omega_1, \omega_2, \dots\}$
- $\mathrm{cof}(\omega_{42}) = ?\ \aleph_{42}$

Sup Sia κ un cardinale infinito $\mathrm{cof}^{V.1}(\kappa) = \mathrm{cof}^{V.2}(\kappa)$ (λ_1, λ_2)

Dim

$(\lambda_1 \leq \lambda_2)$ Sia A cofinale in κ.

$$\kappa = \sup_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \alpha$$

$$\kappa = \left| \bigcup_{\alpha \in A} \alpha \right| \leq \sum_{\alpha \in A} |\alpha| \leq \sum_{\alpha \in A} \kappa \leq |A| \cdot \kappa \leq \kappa$$

($|\alpha| < \kappa$)

$(\lambda_2 \leq \lambda_1)$

$$\kappa = \sum_{i \in \lambda_1} \kappa_i = |\lambda_1| \cdot \sup\{\kappa_i \mid i \in \lambda_1\}$$

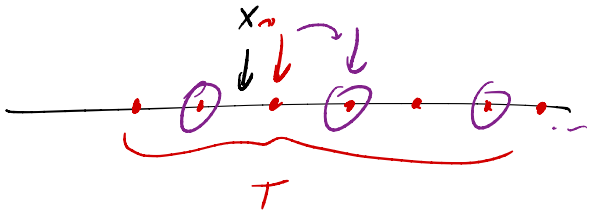
($\kappa_i < \kappa$; $|\lambda_1|$: κ operazioni; $\sup\{\kappa_i \mid i \in \lambda_1\}$: κ)

∎

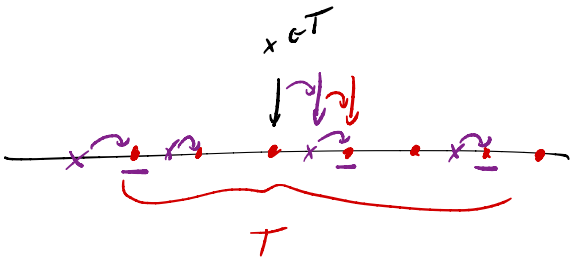
Oss Sia $(S, <_S)$ b.o. ord. $T \subseteq S$ cofinale in S.
Indichiamo con $<_T$ la relazione $<_S \cap (T \times T)$.
$\operatorname{cof}(S, <_S) = \operatorname{cof}(T, <_T)$.

Dim

U_α cof. in T



V_x cof. in S



∎

Oss k card. inf. $\rightarrow \operatorname{cof}(k) \leq k$

Dim. $\sum_{i<k} 1 = k$

Oss Sia $(S, <)$ b.o. ord.

$$\operatorname{cof}(\operatorname{cof}(S, <)) = \operatorname{cof}(S, <)$$

Dim ($\leq$) oss. precedente.

Dobbiamo dimostrare $\operatorname{cof}(S, <) \leq \operatorname{cof}(\operatorname{cof}(S, <))$
Per assurdo, assumiamo $\operatorname{cof}(S, <) > \operatorname{cof}(\operatorname{cof}(S, <)) = k$

$$\operatorname{cof}(S, <) = \sum_{i<k} \kappa_i \qquad \kappa_i < \operatorname{cof}(S, <)$$

Sia A cofinale in S t.c. $|A| = cof(S, <)$

Esistono $\{A_i\}_{i<\kappa}$ disgiunti con $|A_i| = \kappa_i$ e

$\bigcup_{i<\kappa} A_i = A$. Per ogni i esiste $x_i \in S$ t.c.

$\forall y \in A_i \quad y < x_i$. $\quad |\{x_i \mid i < \kappa\}| \leq \kappa < cof(S, <)$

quindi $\{x_i \mid i<\kappa\}$ non è cofinale. Di conseguenza

$\exists x \in S \quad \forall i < \kappa \quad x_i < x$.

Ma $\forall a \in A \quad \exists i < \kappa \quad a \in A_i$

$\Downarrow$

$a < x_i < x \quad \rightsquigarrow \Leftarrow$ ☐

Cor (König)

Sia κ card. infinito, λ card ≥ 2.

$$\kappa < \kappa^{cof(\kappa)} \qquad \kappa < cof(\lambda^\kappa)$$

Dim Sia $\mu = cof(\kappa)$ $\quad \kappa = \sum_{i<\mu} \kappa_i \quad \kappa_i < \kappa$.

$$\kappa = \sum_{i<\mu} \kappa_i < \prod_{i<\mu} \kappa = \kappa^\mu = \kappa^{cof(\kappa)}$$

Assumiamo per assurdo $cof(\lambda^\kappa) \leq \kappa$

$$\lambda^\kappa < (\lambda^\kappa)^{cof(\lambda^\kappa)} = \lambda^{\kappa \cdot cof(\lambda^\kappa)} \leq \lambda^{\kappa \cdot \kappa} = \lambda^\kappa \quad \rightsquigarrow \Leftarrow$$ ☐

Def Per κ cardinale infinito

$$\beth(\kappa) = \kappa^{\mathrm{cof}(\kappa)}$$

[ghimel]

Con Per κ cardinale infinito

$$\kappa < \beth(\kappa) \qquad \mathrm{cof}(\kappa) < \mathrm{cof}(\beth(\kappa))$$

Dim

$$\mathrm{cof}\left(\kappa^{\mathrm{cof}(\kappa)}\right) > \mathrm{cof}(\kappa).$$

□

Def: Un cardinale κ è:

- regolare se $\mathrm{cof}(\kappa) = \kappa$
- singolare se $\mathrm{cof}(\kappa) < \kappa$
- successore se $\kappa = \aleph_{\alpha+1}$ per qualche ordinale α
- limite se $\kappa = \aleph_\lambda$ con λ ordinale limite
- limite forte se $\forall \alpha \ \aleph_\alpha < \kappa \to 2^{\aleph_\alpha} < \kappa$
- debolmente inaccessibile se è regolare e limite
- (fortemente) inaccessibile se è regolare e limite forte

$$\mathrm{cof}(\aleph_{\omega_2}) = \aleph_{\omega_2} \qquad \wedge \qquad \vee$$

$$\mathrm{cof}(\aleph_\omega) = \aleph_0$$

