④ Assioma del paio

"dati a, b esiste $\{a, b\}$"

- $\forall a \, \forall b \, \exists P \, \forall x \quad x \in P \leftrightarrow (x = a \vee x = b)$

$\forall$ dati $\exists S \quad \forall x \quad x \in S \leftrightarrow$ proprietà(dati, x)

Prop. $\forall a \, \forall b \, \exists! P \, \forall x \quad x \in P \leftrightarrow (x = a \vee x = b)$

Dim. Supponiamo di avere P_1, P_2 tali che

$\forall x \quad x \in P_1 \leftrightarrow (x = a \vee x = b)$

$\forall x \quad x \in P_2 \leftrightarrow (x = a \vee x = b)$

$x \in P_1 \leftrightarrow (x = a \vee x = b) \leftrightarrow x \in P_2$

$\hookrightarrow$ estensionalità $\Rightarrow P_1 = P_2$ ∎

Notazione

$P = \{a, b\} \overset{\text{def}}{\equiv} \forall x \; x \in P \leftrightarrow (x = a \vee x = b)$

Prop $\forall a \, \exists! S \, \forall x \quad x \in S \leftrightarrow x = a$

Dim Fisso $b = a$ nell'assioma

$\exists! S \quad \forall x \quad x \in S \leftrightarrow \underbrace{(x = a \vee x = a)}_{x = a}$

∎

Oss. $\{a, b\} = \{b, a\}$

Dim. Passo un qualunque x

$x \in \{a, b\} \overset{\text{def.}}{\iff} x = a \vee x = b$

$x = b \vee x = a \overset{\text{def.}}{\iff} x \in \{b, a\}$ ☒

Def (coppia di Kuratowski)

$(a, b) \overset{\text{def.}}{\equiv} \{\{a\}, \{a, b\}\}$

Prop.

$\forall a \; \forall b \; \forall a' \; \forall b' \quad (a, b) = (a', b') \leftrightarrow (a = a' \wedge b = b')$

$\forall a, b, a', b'$

Dim. Data $c = (a, b)$ cerchiamo formule Φ e Ψ t.c.

$\Phi(c, x) \leftrightarrow x = a \qquad \Psi(c, y) \leftrightarrow x \in b$

$(a, b) \overset{\text{def.}}{\equiv} \{\{a\}, \{a, b\}\} = c$

$x = a \leftrightarrow \forall y \in c \quad x \in y$

$\rightarrow \quad y \in c \iff y = \{a\} \vee y = \{a, b\}$

$\Downarrow$

$a \in y \vee a \in y$

$\leftarrow$ Osserviamo che, in particolare, $\{a\} \in c$

$\forall y \in c \quad x \in y \implies x \in \{a\} \implies x = a$

Passiamo a b

- Caso 1. $\exists z \quad c = \{z\}$

$$\exists z \quad c = \{z\} \implies \{a\} = \{a, b\} \implies b = a$$

- Caso 2 ~~$\exists z$~~ $\quad c = \{z\}$

$$t = \{a, b\} \leftrightarrow (t \in c \wedge t \neq \{a\})$$

$$x = b \leftrightarrow (x \in \{a, b\} \wedge x \neq a).$$

Def

$$(a, b, c) \overset{\text{def}}{\equiv} ((a, b), c)$$

$$(a, b, c, d) \equiv ((a, b, c), d)$$

"$(a_1 \dots a_{n+1}) \equiv ((a_1 \dots a_n), a_{n+1})$"

P-or.

$$(a, b, c) = (a', b', c') \leftrightarrow (a = a' \wedge b = b' \wedge c = c')$$

Esercizio facile.

(5) Assioma dell'unione

"Dato un insieme A esiste l'unione B di tutti i suoi elementi"

"Dato A esiste B i cui elementi sono tutti e soli gli elementi degli el. di A"

$$\forall A \; \exists B \; \forall x \; x \in B \leftrightarrow \exists y \in A \; x \in y$$

Pro.

$$\forall A \; \exists! B \; \forall x \; x \in B \leftrightarrow \exists y \in A \; x \in y$$

Notazione

$$B = \bigcup A \overset{\text{def}}{\equiv} \forall x \; x \in B \leftrightarrow \exists y \in A \; x \in y$$

Prop. $\forall a, b \; \exists! c \; \underbrace{\forall x \; x \in c \leftrightarrow (x \in a \vee x \in b)}_{c = a \cup b}$

Dim. Sia $c = \bigcup \{a, b\}$

$$x \in c \iff \exists \underbrace{y \in \{a, b\}}_{y = a \vee y = b} \; x \in y$$

$$\iff \exists y \; (y = a \vee y = b) \wedge x \in y$$

$$\iff x \in a \vee x \in b$$

Def Sia C una classe non vuota.

$$B = \bigcap C \overset{\text{def.}}{\equiv} \forall x \; x \in B \leftrightarrow \forall y \in C \; x \in y$$

$$\forall y \in \emptyset \quad x \in y$$

$$\forall y \; (y \in \emptyset) \Rightarrow (x \in y)$$

F

V

Prop La definizione è ben data

Dim Devo dim. che B esiste.

Fisso un qualunque $k \in C$. Considero

$$B \overset{\text{def.}}{=} \{ x \in k \mid \forall y \in C \; x \in y \}$$

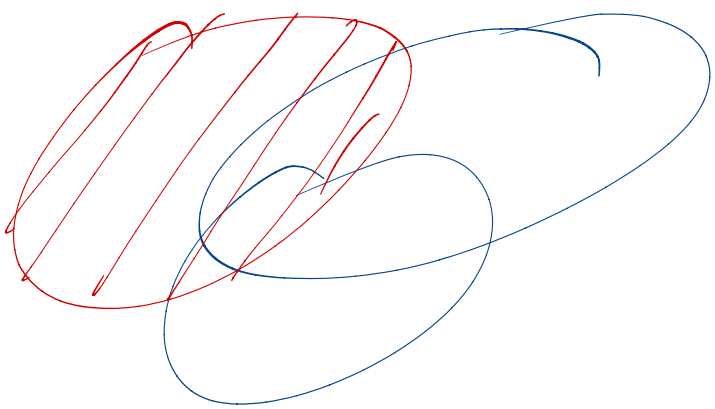
$x \in B \rightarrow \forall y \in C \; x \in y$ immediata

$\forall y \in C \; x \in y \rightarrow x \in B$

$\hookrightarrow x \in k \rightarrow x \in k \wedge \forall y \in C \; x \in y$

$\Downarrow$

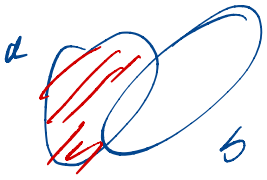
$x \in B.$



Def $a \cap b \overset{\text{def.}}{=} \bigcap \{a, b\}$

Prop. $x \in a \cap b \hookrightarrow (x \in a \wedge x \in b)$

Dim Esercizio.



Prop. $a \setminus b \overset{\text{def.}}{=} \{x \in a \mid x \notin b\}$

Proprietà (alcune) di $\cup$ $\cap$ $\setminus$

$$a \cup b = b \cup a$$

$$a \cap b = b \cap a$$

$$a \cup (b \cup c) = (a \cup b) \cup c$$

$$\cap \quad \cap \qquad\qquad \cap \quad \cap$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

$$\cap \quad \cup \qquad\qquad \cap \quad \cup \quad \cap$$

$$a \setminus (b \cup c) = (a \setminus b) \cap (a \setminus c)$$

$$\cap \qquad\qquad \cup$$

$$\bigcup \{\{a\}, \{a, b\}\} = \{a, b\}$$

$$\bigcup (\emptyset, \emptyset)$$

$$\shortparallel$$

$$\bigcup \{\{\emptyset\}, \underbrace{\{\emptyset, \emptyset\}}_{\{\emptyset\}}\} = \bigcup \underbrace{\{\{\emptyset\}, \{\emptyset\}\}}_{\{\{\emptyset\}\}} =$$

$$= \bigcup \{\{\emptyset\}\} = \{\emptyset\}$$

Dim

$$a \setminus (b \cup c) = (a \setminus b) \cap (a \setminus c)$$

$$x \in a \setminus (b \cup c) \iff x \in a \wedge \neg (x \in b \cup c)$$

$$x \in a \wedge \underbrace{\neg (x \in b \vee x \in c)}$$

$$x \in a \wedge \overbrace{x \notin b \wedge x \notin c}$$

$$x \in a \wedge x \notin b \wedge x \in a \wedge x \notin c$$

$$(x \in a \wedge x \notin b) \wedge (x \in a \wedge x \notin c)$$

$$x \in a \setminus b \qquad x \in a \setminus c$$

$$x \in (a \setminus b) \cap (a \setminus c)$$

Def $\{a, b, c\} \overset{def.}{\equiv} \{a\} \cup \{b\} \cup \{c\}$

"$\{a_1 \dots a_n\} \equiv \{a_1\} \cup \dots \cup \{a_n\}$"

Prop.

$x \in \{a, b, c\} \leftrightarrow (x = a \vee x = b \vee x = c)$

Es.

(6) Assioma delle parti

"Dato A esiste l'insieme dei sottoins. di A"

$\forall A \; \exists! B \quad \forall x \quad x \in B \leftrightarrow x \subseteq A$

Def.

$B = \mathcal{P}(A) \overset{def.}{\equiv} \forall x \quad x \in B \leftrightarrow x \subseteq A$

Def.

$C = A \times B \overset{def.}{\equiv} \forall x \; x \in C \leftrightarrow \exists a \in A \; \exists b \in B \; x = (a, b)$

$C = \{(a, b) \mid a \in A \quad b \in B\}$

Prop. $\forall A \; \forall B \; \exists! \; A \times B$

Dim

Fissiamo $a \in A$ $b \in B$

$x = (a,b) \iff x = \{\{a\}, \{a,b\}\}$

$\{a,b\} \subseteq A \cup B$

$\in \mathcal{P}(A \cup B)$

$\subseteq \mathcal{P}(A \cup B)$

$\in \mathcal{P}(\mathcal{P}(A \cup B))$

$C = \{x \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \exists a \in A \ \exists b \in B \ \ x = (a,b)\}$

Devo dimostrare "$C = A \times B$"

$x \in C \hookrightarrow \exists a \in A \ \exists b \in B \ \ x = (a,b)$

$\rightarrow$ Immediata

$\leftarrow$ Fissiamo $a \in A$, $b \in B$ t.c. $x = (a,b)$

$\ldots \Rightarrow x \in \mathcal{P}(\mathcal{P}(A \cup B))$

$\Rightarrow x \in C$ ∎

Def. Una relazione R tra A e B è

$R \subseteq A \times B$

$x \, R \, y \equiv (x, y) \in R$