Teo (Cantor – Bernstein)

$$|A| \leq |B| \wedge |B| \leq |A| \rightarrow |A| = |B|$$

Ossia se esistono

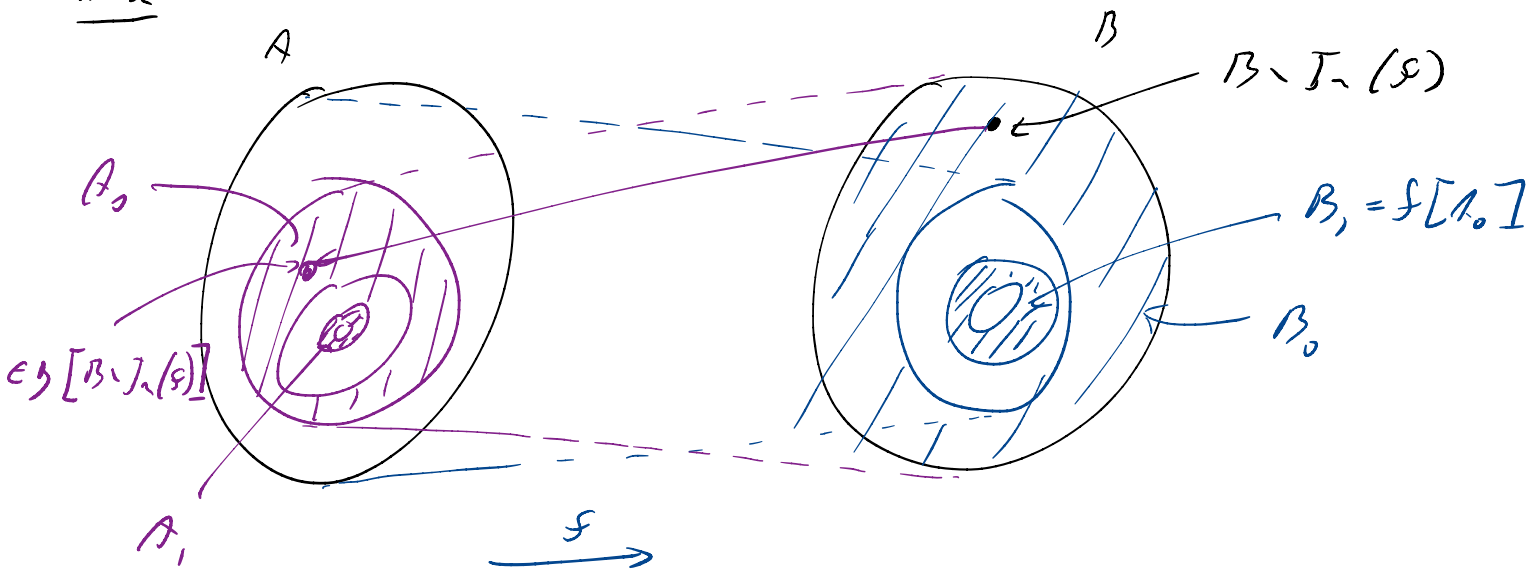
$$f : A \rightarrow B \quad \text{INIETTIVA}$$

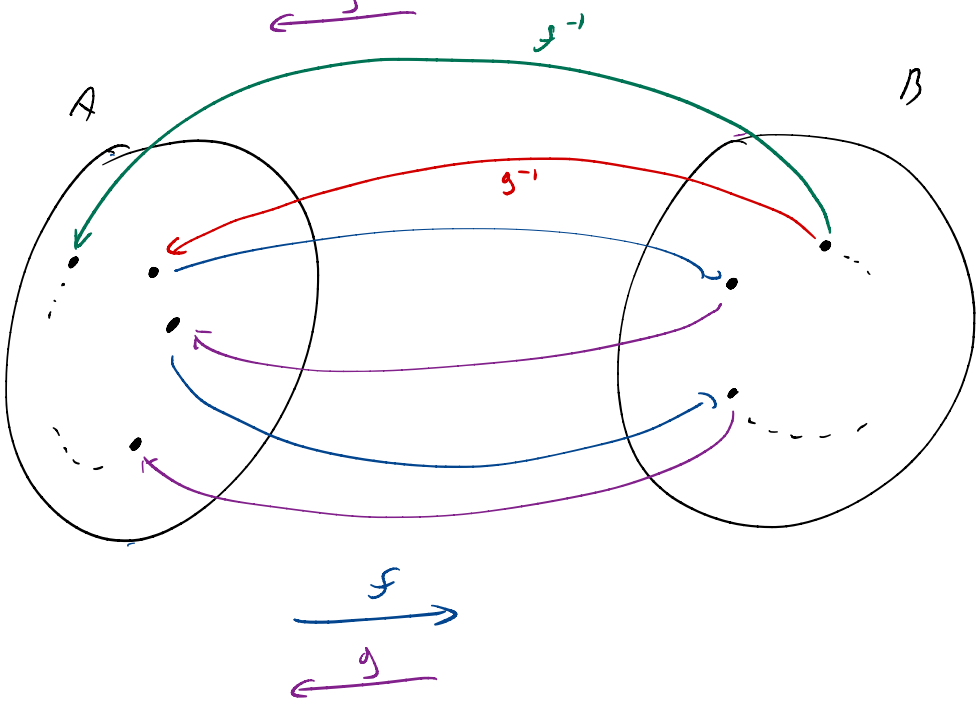
$$g : B \rightarrow A$$

Allora esiste

$$h : A \rightarrow B \quad \text{BIUNIVOCA}$$

Dim





$\xrightarrow{c} f^{-1}(g^{-1}(a)) \xrightarrow{f} g^{-1}(a) \xrightarrow{b} a \xrightarrow{f} f(a) \xrightarrow{c} g(f(a)) \xrightarrow{f} f(g(f(a))) \xrightarrow{c}$

$B_0 = B \setminus Im(f)$ $\quad A_i = g[B_i]$ $\quad B_{s(i)} = f[A_i]$

$$\omega \longrightarrow \mathcal{P}(B) \times \mathcal{P}(A)$$

$$i \longmapsto (B_i, A_i)$$

$$B_* = \bigcup_{i \in \omega} B_i \overset{def.}{=} \bigcup \{B_i \mid i \in \omega\}$$

$$A_* = \bigcup_{i \in \omega} A_i$$

$h : A \to B$ $\qquad$ $k : B \to A$

$$a \longmapsto \begin{cases} g^{-1}(a) & \text{se } a \in A^* \\ f(a) & \text{altrimenti} \end{cases} \qquad b \longmapsto \begin{cases} g(b) & \text{se } b \in B_* \\ f^{-1}(b) & \text{altrimenti} \end{cases}$$

$b \notin B_* \Rightarrow b \notin B_0 = B \setminus Im(f) \Rightarrow b \in Im(f)$.

Verifica: $k \circ h = Id_A$ ossia $k \circ h(a) = a$

caso $a \in A_*$. $\quad a \in A_i = g[B_i]$ quindi $a = g(b)$ $\quad \exists b \in B_i$

$k(h(a)) = k(g^{-1}(a)) = k(b) = g(b) = a$

caso $a \notin A_*$ $\quad$ Oss. $f(a) \notin B_*$

Per assurdo $f(a) \in B_i \to i \neq 0 \to i = s(j)$

$f(a) \in B_{S(f)} = f[A_f]$

$a \in A_f \rightsquigarrow$

$h(h(a)) = h(f(a)) = f^{-1}(f(a)) = a$

$\square$

Def. $|A| < |B| \overset{\text{def.}}{\equiv} |A| \leq |B| \wedge |A| \neq |B|$

Es. $|\mathbb{Z}| \neq |\mathbb{N}|$.

Teo. $|A| < |\mathcal{P}(A)|$

Dim. $|A| \leq |\mathcal{P}(A)|$

$f : A \longrightarrow \mathcal{P}(A)$

$a \longmapsto \{a\}$ è iniettiva

$|A| \neq |\mathcal{P}(A)|$

Supponiamo per assurdo che esista

$f : A \to \mathcal{P}(A)$ biiettiva

$f(x) = \{a \in A \mid a \notin f(a)\} \in \mathcal{P}(A)$

$\exists x \in A \ldots$

$x \in S(x) \longrightarrow x \notin S(x) \rightsquigarrow$

Operazioni aritmetiche sulla cardinalità

Def Siano A, B

$$|A| + |B| = |A \sqcup B| \overset{\text{def.}}{=} |A \times \{0\} \cup B \times \{1\}|$$

$$\begin{array}{ccc} \| & \| & \| \\ |A'| + |B'| & = & |A' \sqcup B'| \end{array}$$

$(A \times \{0\}) \cap (B \times \{1\}) = \emptyset$

Oss. $|X| = |X \times \{\clubsuit\}|$

$x \longmapsto (x, \clubsuit)$

$$|A| \cdot |B| = |A \times B|$$

$$|A|^{|B|} = |{}^B A|$$

${}^B A \to {}^{A'} A'$

$b \mapsto b' : B' \to A'$

$b' = f \circ b \circ g^{-1}$

Oss. Le op. sono ben definite

$$|A| + |B| = |A \sqcup B| \overset{\text{def.}}{=} |A \times \{0\} \cup B \times \{1\}|$$

$$\begin{array}{ccc} \| & \| & \| \\ |A'| + |B'| & = & |A' \sqcup B'| \end{array}$$

$f : A \to A'$ $\quad g : B \to B'$ biiettive

$h : A \sqcup B \to A' \sqcup B'$

$(a, 0) \mapsto (f(a), 0)$

$(b, 1) \mapsto (g(b), 1)$

Notazione Per $n \in \omega$ $\quad n \overset{\text{def}}{\equiv} |n|$

Prop. Le operazioni fra cardinalità godono delle proprietà seguenti: denotando, per brevità, con α, β, γ i simboli $|A|, |B|, |C|$

$$\alpha + 0 = \alpha \qquad \alpha + \beta = \beta + \alpha \qquad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

$$\alpha \cdot 0 = 0 \qquad \alpha \cdot \beta = \beta \cdot \alpha \qquad \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$$

$$\alpha \cdot 1 = \alpha \qquad \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

$$\alpha^0 = 1 \qquad (\alpha^\beta)^\gamma = \alpha^{\gamma \cdot \beta} \qquad (\alpha \cdot \beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma$$

$$\alpha^1 = \alpha \qquad \alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$$

$$1^\alpha = 1$$

Dim $\quad |A| \cdot (|B| + |C|) = |A| \cdot |B| + |A| \cdot |C|$

$A \times (B \sqcup C)$

$=$

$A \times (B \times \{0\} \cup C \times \{1\})$

$=$

$A \times (B \times \{0\}) \cup A \times (C \times \{1\})$

$(a, (b, 0))$ $\qquad$ $(a, (c, 1))$

$A \times B \sqcup A \times C$

$=$

$(A \times B) \times \{0\} \cup (A \times C) \times \{1\}$

$((a, b), 0)$ $\qquad$ $((a, c), 1)$

$$A \times (B \times \{0\}) \longrightarrow (A \times B) \times \{0\}$$
$$(a, (b, 0)) \longmapsto ((a, b), 0)$$

$\{$

∎

Cor. $2^{|A|} = |\mathcal{P}(A)|$

Dim. ${}^{A}\{0,1\} \ni f \longrightarrow \{a \in A \mid f(a)=1\} \subseteq \mathcal{P}(A)$

$\mathcal{P}(A) \ni X \longmapsto \chi_X(a) = \begin{cases} 0 & a \notin X \\ 1 & a \in X \end{cases}$

∎

Cor. Teo. di cantor $\iff |A| < 2^{|A|}$

$|A|^2 = |A|$ per A infinito!

Cardinalità finite

Def. A è finito se $\exists n \in \omega \quad |A| = |n|$

infinito altrimenti.

Def (finitezza secondo Dedekind)

A è Dedekind-finito se non può essere messo in corrispondenza biunivoca con un suo sottoinsieme proprio. $\nexists B \subsetneq A \quad |B| = |A|$

$\forall B \subsetneq A \quad |B| < |A|$

Rem. Finito $\to$ Dedekind-finito

Cor $\forall m, n \in \omega \quad m \neq n \rightarrow |m| \neq |n|$

$<$

$\in \longrightarrow |m| < |n|$

$A = n$

Dim: Sia $|A| = |n|$ Sia $B \subseteq A$ con $|B| = |A|$

Vogliamo dimostrare che $B = A$.

Dimostriamo che SPDG, $A = n$.

$f : A \rightarrow n$ biiettiva

Considero $f[B] \subseteq n$ $|n| = |A| = |B| = |f[B]|$

$\rightsquigarrow f[B] = n \quad x \in A \setminus B \quad f(x) \in n \setminus f[B]$

—

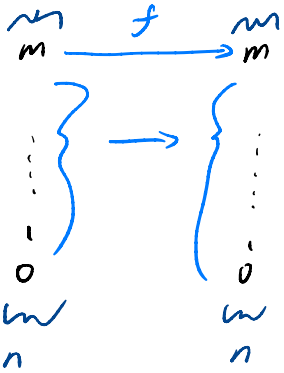
now $B \subseteq n$ $f : n \rightarrow B$ biiettiva.

voglio ottenere $B = n$

Per ind. su n.

$n = \emptyset$ Banale

$n = S(m)$



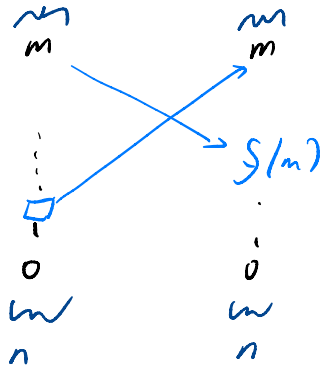
$f|_m : m \rightarrow m$ biiettiva

$\downarrow$

$I_n(f|_m) = m$

$B = I_n(f) = I_n(f|_m) \cup \{m\} =$

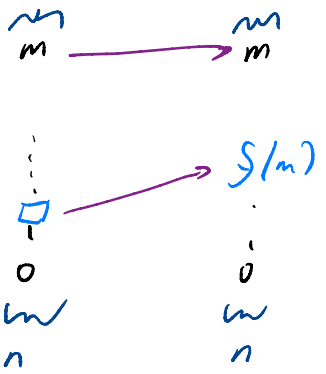
$= m \cup \{m\} = S(m)$



$m \in \mathrm{In}(f)$

$\mathrm{In}(f|_m) \subseteq m \setminus \{f(m)\}$

$\subsetneq m$ ~ i.p. induction.



$\boxtimes$