$f : B \to C$

Prop: A, B, C insiemi $\quad |B| \leq |C|$

$$|A| + |B| \leq |A| + |C| \qquad |A|^{|B|} \leq |A|^{|C|}$$

$$|A| \cdot |B| \leq |A| \cdot |C| \qquad |B|^{|A|} \leq |C|^{|A|}$$

Dim

$$|\underbrace{f[B]}_{B'}| = |B| \qquad B' \subseteq C$$

$$|A| + |B'| \leq |A| + |C|$$

$$\underline{A \times \{0\}} \cup \underline{B' \times \{1\}} \subseteq \underline{A \times \{0\}} \cup \underline{C \times \{1\}}$$

$\subseteq$

Oss $\quad |A \cup B| \leq |A| + |B|$

$f : A \cup B \to A \times \{0\} \cup B \times \{1\}$

$$f(x) = \begin{cases} (x, 0) & \text{se } x \in A \\ (x, 1) & \text{altrimenti} \end{cases}$$

□

Supponiamo che $\quad 2^{\aleph_0} > \aleph_0$

$\Downarrow$

$\square \neq 0, 1 \to \square^{\aleph_0} > \aleph_0$

Prop con $n \in \omega \setminus \{0\}$

$$\aleph_0 + \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0^n = \aleph_0$$

Dim Basta dire che $\aleph_0 \cdot \aleph_0 = \aleph_0$

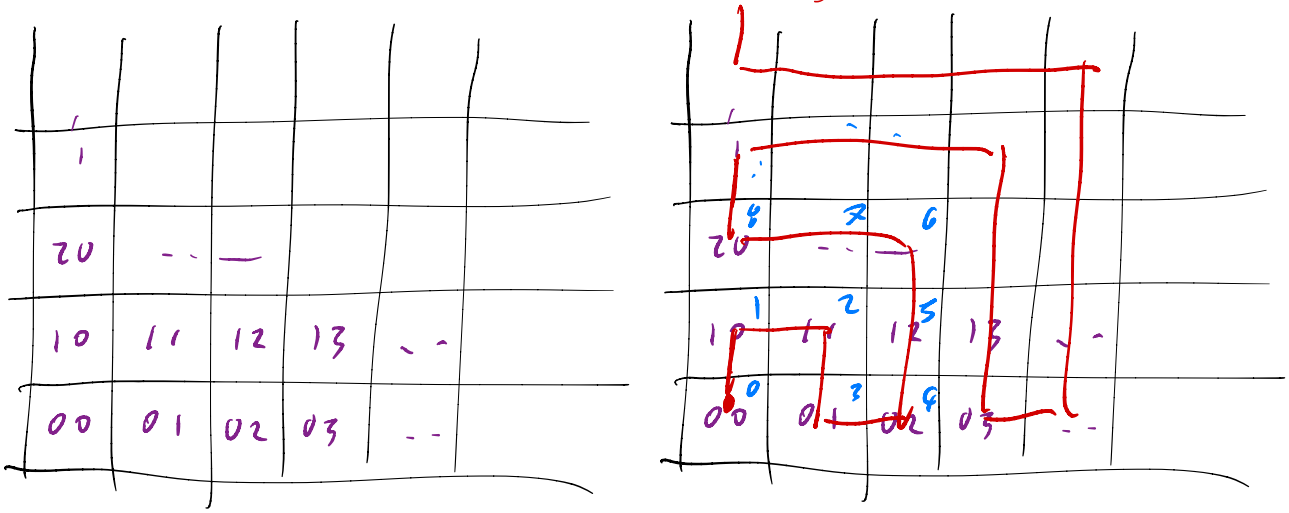
$$\aleph_0 = \aleph_0 + 0 \leq \aleph_0 + \aleph_0 = \aleph_0 \cdot 2 \leq \aleph_0 \cdot \aleph_0 = \aleph_0$$

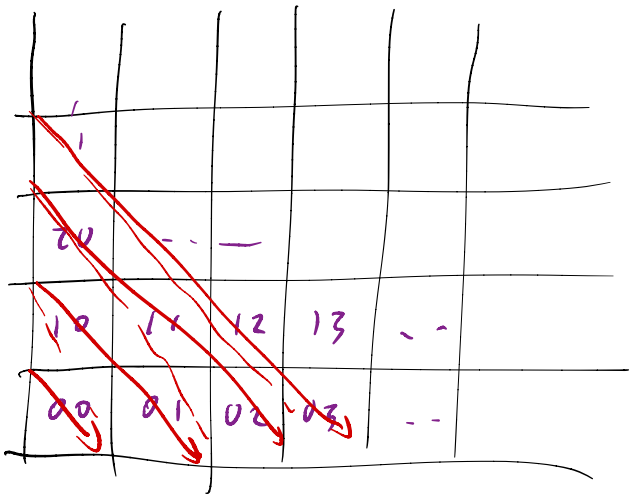
Induzione:

$$\aleph_0^1 = \aleph_0 \qquad \aleph_0^{n+1} = \aleph_0^n \cdot \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0$$

Lemma $\aleph_0 \cdot \aleph_0 = \aleph_0$

$\omega \times \omega$ è in biezione con ω





$$\omega \times \omega \longrightarrow \omega$$

$$(x, y) \longmapsto 2^x(2y+1) - 1$$

$$\omega \times \omega \longrightarrow \omega$$

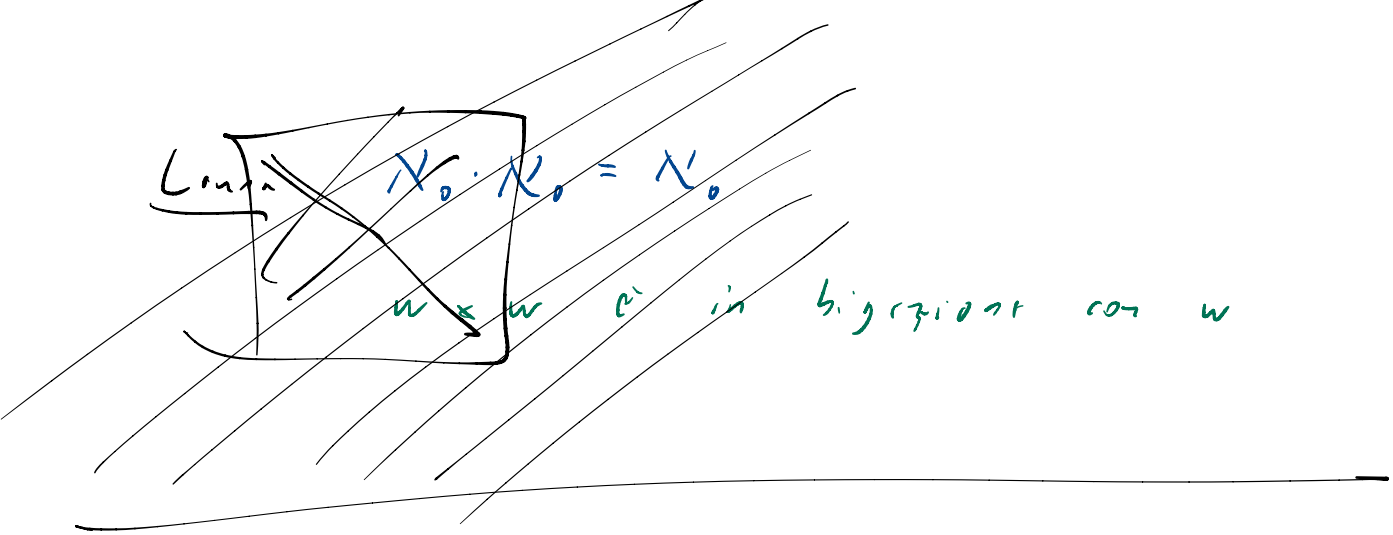
$$(x, y) \longmapsto \frac{(x+y)^2 + 3x + y}{2}$$

$x = 47$ $\qquad$ $y = 1782$

1074872

$$|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

(A infinite $\qquad |A \times A| = |A|$)



Lemma $\aleph_0 \cdot \aleph_0 = \aleph_0$

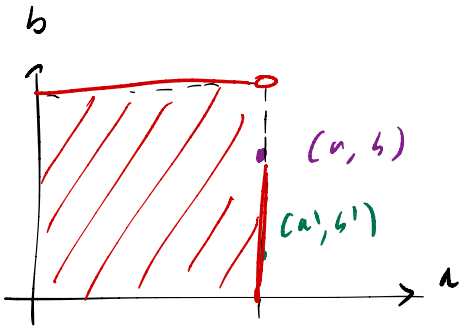
$w \times w$ è in biiezione con w

Dim.

$$(a,b) \prec (a',b') \equiv \max(a,b) < \max(a',b')$$

$$\vee \left(\max(a,b) = \max(a',b') \wedge a < a' \right)$$

$$\vee \left(\max(a,b) = \max(a',b') \wedge a = a' \wedge b < b' \right)$$



$$(a,b) \prec (a',b') \prec (a'',b'')$$

$$\downarrow$$

$$(a,b) \prec (a'',b'')$$

Oss. $(w \times w)_{(a,b)} \equiv \{ x \in w \times w \mid x \prec (a,b) \}$

è finito Sia $n = \max(a,b)$

$$(w \times w)_{(a,b)} \subseteq S(n) \times S(n)$$

$\emptyset_{1,7} \rightarrow$ $A \subseteq \omega \times \omega$ sup. lim de (a,b)

$\downarrow$

A ha un mas.

$\forall \rightarrow A \in (\omega \times \omega)_{(a+1,\, b+1)}$

Sea $A \subseteq \omega \times \omega$ $A \neq \emptyset \rightarrow (a,b) \in A$.

Consideramos $A \cap (\omega \times \omega)_{(a+1,\, b+1)}$ ∎

Def. ($\mathbb{Z}$) $\quad (a,b) \approx a - b$

$(a,b) \sim_{\mathbb{Z}} (a',b') \equiv a + b' = b + a'$

$\omega \times \omega$

$[(a,b)] + [(a',b')] = [(a+a',\, b+b')]$

$-[(a,b)] = [(b,a)]$

$[(a,b)] \cdot [(a',b')] = [(aa' + bb',\, ab' + ba')]$

Def ($\mathbb{Q}$) $\mathbb{Z} \times (\omega \setminus \{0\})$

$(n,d) \sim_{\mathbb{Q}} (n',d') \equiv nd' = dn'$

$\frac{n}{d}$.

Def. $\frac{n}{d}$ è positivo se $n, d \in \omega \setminus \{0\}$

$a < b$ se $b - a$ è positivo.

Esercizio $\forall a, b \in \mathbb{Q}\ (a < b) \to \exists c \in \mathbb{Q}\ a < c < b$

$g : A \to \omega$ iniettiva

Es. $|A| \leq \aleph_0$ $f : A \to B$ suriettiva

allora $|B| \leq \aleph_0$

Def. $h : B \to \omega$

$b \mapsto \min\left(g\left[\{a \in A \mid f(a) = b\}\right]\right)$

$\dfrac{h(b) = h(b')}{g(a) = g(a')}$ con $f(a) = b$ $f(a') = b'$

$a = a' \to \underline{b} = f(a) = f(a') = \underline{b'}$

□

Esempio ($\mathbb{Z}$ è numerabile)

$$\omega \times \omega \longrightarrow \mathbb{Z}$$

$$(a, b) \longmapsto [(a, b)]_{\mathbb{Z}} \quad \text{suriettiva.}$$

$$\omega \longrightarrow \mathbb{Z}$$

$$n \longmapsto [(n, 0)]_{\mathbb{Z}} \quad \text{iniettiva}$$

Esempio ($\mathbb{Q}$ è numerabile)

$$\mathbb{Z} \times (\omega \setminus \{0\}) \longrightarrow \mathbb{Q}$$

$$(a, b) \longmapsto [(a, b)]_{\mathbb{Q}} \quad \text{suriettiva.}$$

$$\omega \longrightarrow \mathbb{Q}$$

$$n \longmapsto [(n, 1)]_{\mathbb{Q}} \quad \text{iniettiva}$$

$\uparrow$

$\frac{n}{1}$

□

Vorremo dire che se $|A| \leq \aleph_0$ e $\forall a \in A \ |a| \leq \aleph_0$ allora $|\bigcup A| \leq \aleph_0$.

"Tecnicismo di dimostrazione"

Cerco una $h : \omega \times \omega \longrightarrow \bigcup A$ suriettiva.

Prendo $f : \omega \to A$ surgettiva.

Per ogni $f(i) \in A$ supponiamo che $|f(i)| \leq \aleph_0$,

quindi prendo $g_i : \omega \to f(i)$ surgettiva.

$$h : \omega \times \omega \to \bigcup A$$
$$(i, j) \mapsto g_i(j)$$

Prop Sia $A = \{a_i \mid i \in \omega\}$

Sia $\{f_i\}_{i \in \omega}$ una successione di enumerazioni di a_i

$f_i : \omega \to a_i$ surgettiva

Allora $\bigcup A = \bigcup_{i \in \omega} a_i$ è al più numerabile.

Dim

$$h : \omega \times \omega \to \bigcup A$$
$$(i, j) \mapsto f_i(j).$$

Prop: $|A| \leq \aleph_0 \to |\mathcal{P}^{fin}(A)| \leq \aleph_0$

$\{X \in \mathcal{P}(A) \mid |X| < \aleph_0\}$

Esempio di applicazione

$$|\mathbb{Z}[x]| = \aleph_0$$

polinomi in x a coeff. in $\mathbb{Z}$.

$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ con $a_0 \dots a_n \in \mathbb{Z}$.

$\{(0, a_0)\ (1, a_1) \dots (n, a_n)\}$

$\mathbb{Z}[x] \subseteq \mathcal{O}^{fin}(\omega \times \mathbb{Z})$.

$|\mathbb{Z}[x]| \leq \aleph_0$ $\qquad$ $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$

$a_0 \mapsto a_0$

$\subseteq \mathbb{R}$

Es. $|\overline{\mathbb{Q}}^{alg}| = \aleph_0$

$\mathbb{Q}_n^{alg} = \bigcup_{i \in \omega} A_i$

$\{x \in \mathbb{R} \mid x$ è radice di un polinomio di grado $i\}$

$f_i : \omega \to$ pol. di grado i suriettiva

$g_i : w \times w \to A_i$

$(x,y) \mapsto$ y-esimo zero di $f_i(x)$ se $\exists$

altrimenti 0

Prop: $|A| \leq \aleph_0 \to |\mathcal{P}^{fin}(A)| \leq \aleph_0$

$\{X \in \mathcal{P}(A) \mid |X| < \aleph_0\}$

Dim $\mathcal{P}^{\leq n}(A) = \{X \in \mathcal{P}(A) \mid |X| \leq n\}$

$\mathcal{P}^{fin}(A) = \bigcup_{n \in w} \mathcal{P}^{\leq n}(A)$

$f : w \to w \times A$ suriettiva

$g_n : w \to \mathcal{P}^{\leq n}(A)$ suriettiva

$g_0 : w \to \{\emptyset\}$

g_{n+1} $x \mapsto$ $\emptyset$ se $x = 0$

$g_n(f_1(x-1)) \cup \{f_2(x-1)\}$ $x > 0$