Teo $|\mathbb{R}| = 2^{\aleph_0} = |\mathcal{P}(\omega)|$

$\geq$

$f : \mathcal{P}(\omega) \longrightarrow \mathbb{R}$

$S \longmapsto 0.\, a_1^S\, a_2^S\, a_3^S\, a_4^S \ldots$

$a_i = \begin{cases} 1 & \text{se } i \in S \\ 0 & \text{se } i \notin S \end{cases}$

$f(S) = f(T) \iff \forall i \;\; a_i^S = a_i^T$

$\forall i \;\; i \in S \Leftrightarrow i \in T$

$S = T$

$0.\, a_0\, a_1\, a_2\, a_3 \ldots = \sum_{i=0}^{\infty} a_i \cdot 10^{-i}$

$\downarrow$

$\sup \left\{ \sum_{i=0}^{n} a_i \cdot 10^{-i} \;\middle|\; n \in \mathbb{N} \right\}$

Prop. Sia $(A, <)$ completo senza estremi. Sia $B \subseteq A$ con $|B| = \aleph_0$ e B denso in A. Allora $(A, <)$ è isomorfo a $\mathbb{R}$.

Dim Sia $\tilde{A}$ l'insieme delle sezioni su B.

$f : A \to \tilde{A} \qquad f(a) = \{x \in B \mid x < a\}$

$g : \tilde{A} \to A \qquad g(X) = \sup X$

Verifica $g(f(a)) = a$

$$\sup \{ x \in B \mid x < a \} \stackrel{?}{=} a$$

Sia $y < a$ $\quad \exists z \in B \quad y < z < a$

f è un iso. d'ordine fra $(A, \leq)$ e $(\tilde{A}, \subseteq)$

$$f(a) = \{ x \in B \mid x < a \}$$

$a < b \Rightarrow x < a \to x < b \Rightarrow f(a) \subseteq f(b)$

Def $\quad g : B \to \mathbb{R}$ iso.

$$\tilde{g} : \tilde{A} \to \mathbb{R}$$

$$X \mapsto g[X]$$

∎

Def 1 $\quad$ Def: $x, y \subseteq \mathbb{R}$

$$x + y = \{ a + b \mid a \in x \wedge b \in y \}$$

Se $x, y > 0$

$$x \cdot y = \{ a \cdot b \mid a \in x \wedge b \in y \wedge a, b > 0 \} \cup \{ q \in \mathbb{Q} \mid q \leq 0 \}$$

Def $(F, 0, 1, +, \cdot, <)$ è un ordinato se

- $(F, <)$ ordine totale
- $(F, 0, 1, +, \cdot)$ campo
- $\forall x, y, z \quad x < y \iff x + z < y + z$
- $\forall x, y > 0 \quad x \cdot y > 0$.

Teo $\mathbb{R}$ è l'unico campo ord. completo a meno di isomorfismi.

Oss $\left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0}$

$$2^{\aleph_0} = \left(2^{\aleph_0}\right)^1 \leq \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$$

Con Siano α, β o "sinistro" o $\aleph_0$ o $2^{\aleph_0}$

$$\alpha + \beta = \alpha \cdot \beta = \max(\alpha, \beta)$$

↑

Occorre $\alpha \cdot 0 = 0 \cdot \beta = 0$.

Rem $\beta \leq \alpha \qquad \aleph_0 \leq \alpha$

$$\alpha = \alpha + 0 \leq \alpha + \beta \leq \alpha + \alpha \leq \alpha \cdot 2$$

$$\alpha = \alpha \cdot 1 \leq \alpha \cdot \beta \leq \alpha \cdot \alpha = \alpha^2 = \alpha$$

(↑ 1)

$$(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$$

$$2^{\aleph_0} \leq (2^{\aleph_0})^2 \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$$

Prop. Se $2 \leq \alpha \leq 2^{\aleph_0}$ $\quad \alpha^{\aleph_0} = 2^{\aleph_0}$

$$2^{\aleph_0} \leq \alpha^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$$

$0 \; 1 \; 2 \; 3 \; \ldots \; \aleph_0 \qquad 2^{\aleph_0}$

qui non c'è niente. ?



0 $\quad 0 \ldots 9$ ⤵

1 $\quad 10 \ldots 19$ ⤵ $\quad$ ↳ 0

2 $\quad 20 \ldots 29$ ⤵ $\quad$ ↳ 1

3 $\quad 30 \ldots 39$ ⤵ $\quad$ ↳ 2

⋮



Oss $|A| = \aleph_0 \quad |B| < \aleph_0 \Rightarrow |A \setminus B| = \aleph_0$

Dim $A = B \cup (A \setminus B)$ ☐

Lemma $|A| = 2^{\aleph_0} \quad |B| \leq \aleph_0 \Rightarrow |A \setminus B| = 2^{\aleph_0}$

Dim

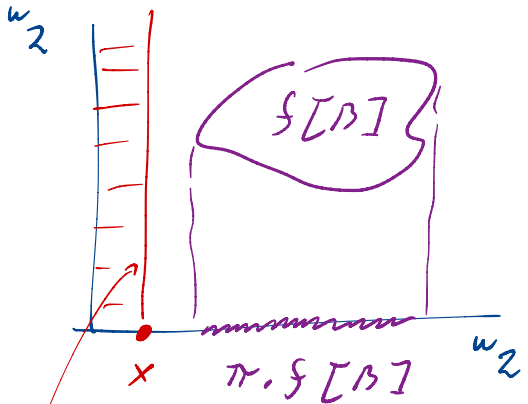
$2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$

$|{}^{\omega}2 \times {}^{\omega}2| = |A|$

$f : A \to {}^{\omega}2 \times {}^{\omega}2$ biunivoca

$|f[B]| \leq \aleph_0$

$|\pi \circ f[B]| \leq \aleph_0$



$|\{x\} \times {}^{\omega}2| = 2^{\aleph_0}$

C

$C \cap f[B] = \emptyset$

$f^{-1}(C) \cap B = \emptyset$ ☐

Dim $(|\mathbb{R}| = 2^{\aleph_0})$

$A = \{X \in \mathcal{P}(\omega) \mid X \neq \emptyset \wedge |\omega \setminus X| \geq \aleph_0\}$

$]0,1[\ni 0.01100111000\ldots$

$\{1, 2, 5, 6, 7, \ldots\}$

① $|A| = 2^{\aleph_0}$

$A = \mathcal{P}(\omega) \setminus (\{\emptyset\} \cup \{X \in \mathcal{P}(\omega) \mid |\omega \setminus X| < \aleph_0\})$

$|\mathcal{P}(\omega)| = 2^{\aleph_0}$ $\quad$ $|\mathcal{P}^{fin}(\omega)| = \aleph_0$

② Dico def. $X, Y \in A$

$X < Y \equiv \exists i \in \omega \; X \cap i = Y \cap i \wedge i \in Y \wedge i \notin X$

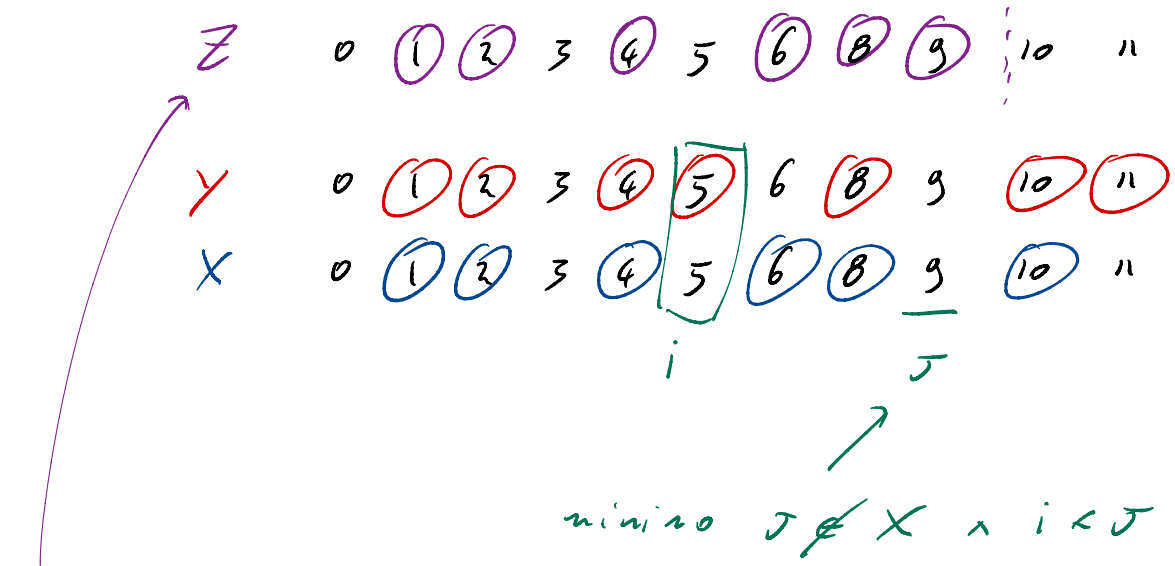
Y 0 1 2 3 4 5 6 8 9 10 11

X 0 1 2 3 4 5 6 8 9 10 11

Sia $i = \min X \triangle Y$ se $i \in Y$ allora $X < Y$

$(X \setminus Y) \cup (Y \setminus X)$ se $i \in X$ allora $Y < X$

Considero $B = \mathcal{P}^{fin}(\omega) \setminus \{\emptyset\} \subseteq A$



$Z = (X \cap \mathcal{J}) \cup \{\mathcal{J}\}$

$b : B \to Q$

$A \longrightarrow$ sezione su $B \longrightarrow$ sezione su Q

$a \longmapsto \{x \in B \mid x < a\} \longmapsto b[\{x \in B \mid x < a\}]$

∎