Con. Esistono le funzioni (classe!) di somma, prodotto e potenza di ordinali così definite:

$$\alpha + 0 = \alpha$$

$$\alpha + (S(\beta)) = S(\alpha + \beta)$$

$$\alpha + \lambda = \sup \{ \alpha + \beta \mid \beta < \lambda \}$$

$$\alpha \cdot 0 = 0$$

$$\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$$

$$\alpha \cdot \lambda = \sup \{ \alpha \cdot \beta \mid \beta < \lambda \}$$

$$\alpha^0 = 1$$

$$\alpha^{S(\beta)} = \alpha^\beta \cdot \alpha$$

$$\alpha^\lambda = \sup \{ \alpha^\beta \mid \beta < \lambda \}$$

Forma normale di Cantor Ha qualunque $\alpha \in Ord$ si scrive in modo unico come

$$\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \dots + \omega^{\beta_n} \cdot k_n$$

con $n \in \omega$, $k_1, k_2, \dots, k_n \in \omega \setminus \{0\}$ e $\beta_1 > \beta_2 > \dots > \beta_n$.

Oss Se $\alpha, \beta \in \omega$ (in ω / ordinale)

$$\alpha + \beta = \alpha + \beta$$

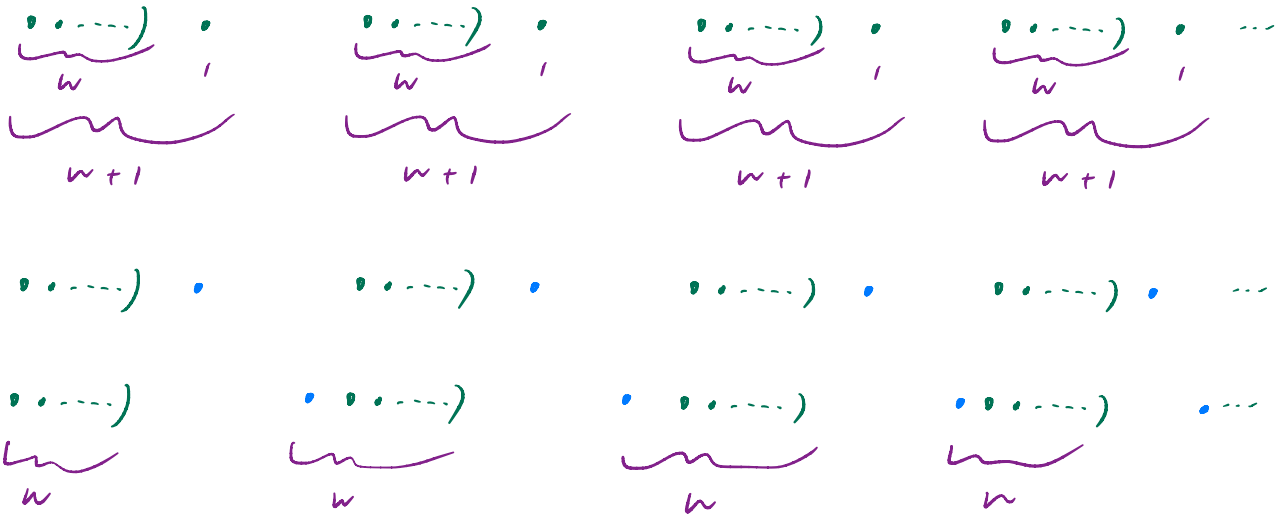
$$\alpha \cdot \beta = \alpha \cdot \beta$$

$$\alpha^\beta = \alpha^\beta$$

$\omega + 1 = \omega^1 \cdot 1 + \omega^0 \cdot 1$ f. normale

$1 + \omega = \sup \underbrace{\{ 1 + n \mid n \in \omega \}}_{\subseteq \omega} \leq \omega \quad \geq \omega \quad \rightarrow = \omega$

$(\omega+1)\cdot\omega$



$(\omega+1)\cdot\omega = \omega\cdot\omega = \underline{\omega^2}$

$(\omega+1)\cdot\omega = \sup\{(\omega+1)\cdot n \mid n<\omega\}$

$\|\,?$

$\omega\cdot n+1 = \underline{\omega^1\cdot n+\omega^0\cdot 1}$ para $n>0$

$(\omega+1)\cdot n = \omega\cdot n+1$

Caso $n=1$. $(\omega+1)\cdot 1 = \omega\cdot 1+1$

Caso $n=s(m)$ $(\omega+1)\cdot s(m) = \underbrace{(\omega+1)\cdot m}_{=}+\omega+1$

$= \omega\cdot m+1+\omega+1$

$= \omega\cdot m+(1+\omega)+1$

$= \underbrace{\omega\cdot m+\omega}+1$

$= \omega\cdot s(m)+1$

$$(\omega+1)\cdot\omega = \sup\{\omega\cdot n+1 \mid n<\omega\}$$

$$= \sup\{\omega\cdot n \mid n<\omega\} = \omega\cdot\omega$$

$$\omega\cdot n < \omega\cdot n+1 < \omega\cdot n+\omega = \omega\cdot s(n)$$

$$\omega^2 = \omega^{s(s(0))} = \omega^{s(0)}\cdot\omega = (\omega^0\cdot\omega)\cdot\omega = (1\cdot\omega)\cdot\omega$$

$$= \omega\cdot\omega$$

$$(\omega+1)\cdot\omega = \omega^2$$

$$(\omega+1)^2 = (\omega+1)\cdot\underbrace{(\omega+1)}_{s(\omega)} = (\omega+1)\cdot\omega + (\omega+1)$$

$$= \omega^2+\omega+1$$

Osservazione

Calcolare $(\omega+1)^n$ con $n\in\omega$.

$$(\omega+1)^\omega = \omega^\omega$$

Prop $\alpha+\beta \simeq (\alpha,<_\alpha)+(\beta,<_\beta)$

$$(\alpha\sqcup\beta, <_+)$$

$$\alpha\cdot\beta \simeq (\alpha,<_\alpha)\cdot(\beta,<_\beta)$$

$$\alpha^\beta \simeq (\alpha,<_\alpha)^{(\beta,<_\beta)}$$

Dim $\alpha+\beta \sim (\alpha, <_\alpha) + (\beta, <_\beta)$

Por inducción transfinita sobre β.

Caso $\beta = 0$. $\quad \alpha + \beta = \alpha$

$$(\alpha, <_\alpha) + (\beta, <_\beta) = (\underbrace{\alpha \sqcup \emptyset}_{\alpha \times \{0\}}, <_+) \sim \alpha.$$

Caso $\beta = s(\gamma)$ Ip. ind. $\alpha + \gamma \sim (\alpha, <_\alpha) + (\gamma, <_\gamma)$

$$\alpha + s(\gamma) = s(\alpha + \gamma)$$

$$(\alpha, <_\alpha) + (s(\gamma), <_{s(\gamma)}) = (\alpha \sqcup (\gamma \cup \{\gamma\}), <_+)$$
$$\sim ((\alpha, <_\alpha) + (\gamma, <_\gamma)) + 1$$
$$\sim (\alpha+\gamma, <_+) + 1 = \alpha+\gamma \cup \{\alpha+\gamma\}$$
$$\sim s(\alpha+\gamma)$$

Caso $\beta = \lambda$ límite. $\quad (\alpha, <_\alpha) + (\delta, <_\delta)$

Por hipótesis inductiva, dado $\delta < \beta$, $\alpha + \delta \sim (\alpha \sqcup \delta, <_+)$

Sea $\gamma \sim \alpha \sqcup \lambda$ digamos $f : \alpha \sqcup \lambda \to \gamma$ isomorfismo.

$$\lambda = \bigcup \{\eta \mid \eta < \lambda\}$$
$$\alpha \sqcup \lambda = \bigcup \{\alpha \sqcup \eta \mid \eta < \lambda\}$$
$$\gamma = f[\alpha \sqcup \lambda] = \bigcup \{f[\alpha \sqcup \eta] \mid \eta < \lambda\}$$
$$\| \qquad \bigcup \{\alpha + \eta \mid \eta < \lambda\} = \alpha + \lambda$$

$\alpha + \beta$

ρ per in. induttiva.

$\alpha \sqcup \beta$ è un s.i di $\alpha \sqcup \lambda$

$f[\alpha \sqcup \beta]$ è un s.i γ

$\quad \| $

$\alpha + \beta$

Prop $\beta < \gamma \rightarrow \alpha + \beta \leq \alpha + \gamma$

$\alpha \leq \gamma \rightarrow \alpha + \beta \leq \gamma + \beta$

$\beta < \gamma \rightarrow \alpha \cdot \beta \leq \alpha \cdot \gamma$ con $\alpha \neq 0$

$\alpha \leq \gamma \rightarrow \alpha \cdot \beta \leq \gamma \cdot \beta$

$\beta < \gamma \rightarrow \alpha^{\beta} \leq \alpha^{\gamma}$ con $\alpha > 1$

$\alpha \leq \gamma \rightarrow \alpha^{\beta} \leq \gamma^{\beta}$

Dim $\beta < \gamma \rightarrow \alpha + \beta < \alpha + \gamma$

Per induzione γ.

Caso $\gamma = 0$. Vero a vuoto

Caso $\gamma = s(\delta)$. Assumo $\beta < s(\delta) \rightarrow \beta \leq \delta$

$$\alpha + \beta \leq \alpha + \delta < s(\alpha + \delta) = \alpha + s(\delta) = \alpha + \gamma$$

$\downarrow$

se $\beta = \delta$ $\quad \alpha + \beta = \alpha + \delta$

se $\beta < \delta$ $\quad \alpha + \beta < \alpha + \delta$

Caso $\gamma = \lambda$ limite. Allora $\beta < \lambda$

$$\alpha + \beta \leq \sup\{\alpha + \delta \mid \delta < \lambda\} = \alpha + \lambda$$

$\beta < \lambda \to s(\beta) < \lambda$ perché $\delta = s(\beta)$

$$\underline{s(\alpha+\beta)} = \alpha + s(\beta) = \alpha + \delta \leq \underline{\alpha + \lambda} \to$$

$$\alpha + \beta \quad < \quad \alpha + \lambda$$

Oss. Dati X, Y insiemi di ordinali, se

$$\forall \alpha \in X \ \exists \beta \in Y \quad \alpha \leq \beta$$

allora $\sup X \leq \sup Y$

Dim. Basta dire che ogni maggiorante di Y è anche maggiorante di X. ∎

$$\alpha \leq \gamma \to \alpha + \beta \leq \gamma + \beta$$

D.d. su β.

$\beta = 0$ — ovvio $\alpha \leq \gamma \to \alpha + 0 \leq \gamma + 0$

$\beta = s(\delta)$ $\alpha + \delta \leq \gamma + \delta$

$$s(\alpha + \delta) = \alpha + s(\delta) \leq \gamma + s(\delta) = s(\gamma + \delta)$$

$$\Box \leq \Delta \to s(\Box) \leq s(\Delta)$$

$$\Box \leq \Delta \leftrightarrow \Box < \underline{s(\Delta)} \leftrightarrow s(\Box) \leq \underline{s(\Delta)}$$

Caso $\beta = \lambda$ limite. $\alpha \leq \gamma$

$$\sup\{\alpha+\delta \mid \delta<\lambda\} = \alpha+\lambda \leq \gamma+\lambda = \sup\{\gamma+\delta \mid \delta<\lambda\}$$

$\leq$

ip. induttiva.

$\square$

Dim

$$(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma) \longrightarrow \alpha\cdot(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma$$

$$(\alpha\cdot\beta)\cdot\gamma = \gamma\cdot(\beta\cdot\gamma) \longrightarrow \alpha^{\beta+\gamma} = \alpha^{\beta}\cdot\alpha^{\gamma}$$

$$(\alpha^{\beta})^{\gamma} = \alpha^{\beta\cdot\gamma}$$

Dim

$$(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$$

Ind. su γ

$$(\alpha+\beta)+s(\gamma) = s((\alpha+\beta)+\gamma) = s(\alpha+(\beta+\gamma))$$

$$\shortparallel$$

$$\alpha+(\beta+s(\gamma)) = \alpha+s(\beta+\gamma)$$

$$(\alpha+\beta)+\lambda = \sup\{(\alpha+\beta)+\delta \mid \delta<\lambda\}$$

$$\shortparallel$$

$$\alpha+(\beta+\delta)$$

$$\alpha+(\beta+\lambda) = \alpha+\sup\{\beta+\delta \mid \delta<\lambda\}$$

$\square$