$$\alpha + \lambda = \sup \{ \alpha + \varepsilon \mid \varepsilon < \lambda \}$$

$$F(\lambda) = \sup F[\lambda]$$

Oss. $X \subseteq \mathrm{Ord}$ sottoinsieme.

$$\sup \{ \alpha + \beta \mid \beta \in X \} = \alpha + \sup X$$

$\square^{\square}$ $\square^{\square}$

Dim

O $\sup X \in X$ o $\sup X \notin X$

$\sup X = \sigma$

$\sup \{ \alpha + \sigma \} = \alpha + \sup X = \alpha + \sigma$

$\Downarrow$

$\sup X \notin X \Rightarrow \sup X$ è limite

Per assurdo se $\sup X = s(\gamma)$

$\Downarrow$

$\exists \delta \in X \quad \underbrace{\gamma < \delta \leq s(\gamma)}_{\delta = s(\gamma) \notin X}$ $\rightsquigarrow$

$$\sup \{ \alpha + \beta \mid \beta \in X \} = \alpha + \sup X$$

$$= \sup \{ \alpha + \beta \mid \beta < \sup X \}$$

$\blacksquare$

Prop $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$

Dim Ci manca il caso γ limite

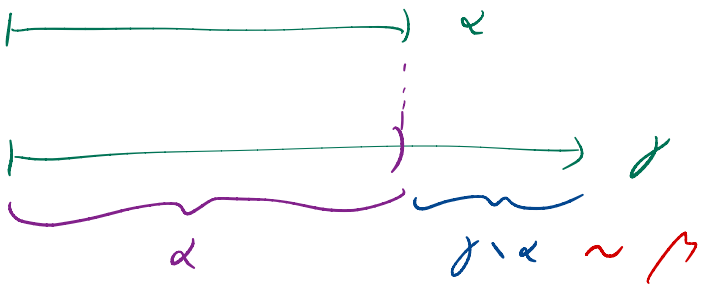
$$(\alpha+\beta)+\gamma \overset{?}{=} \alpha+(\beta+\gamma)$$

$$\sup\{(\alpha+\beta)+\delta \mid \delta<\gamma\} \qquad \alpha+\sup\{\beta+\delta \mid \delta<\gamma\}$$

$$\sup\{\alpha+(\beta+\delta) \mid \delta<\gamma\}$$

$\blacksquare$

Lemma Dati $\alpha \leq \gamma$ esiste un unico β t.c. $\alpha+\beta=\gamma$.



Dim

Unicità: Supponiamo per assurdo $\alpha+\beta=\gamma=\alpha+\beta'$

sppa $\beta<\beta' \Rightarrow \alpha+\beta<\alpha+\beta'$ ↯

Esistenza $\alpha+\beta=\gamma$

Consideriamo il minimo δ T.C. $\alpha+\delta \geq \gamma$.

Caso $\delta=s(\beta)$. $\alpha+\beta \leq \gamma$ (1)

$$s(\alpha+\beta) = \alpha + s(\beta) = \alpha + \delta > \gamma$$

$$\Updownarrow$$

$$\alpha + \beta \geq \gamma \quad (2)$$

$$(1) \wedge (2) \Rightarrow \alpha + \beta = \gamma.$$

Caso δ limite vogliamo escluderlo

$$\alpha + \delta > \gamma. \qquad \sup\{\alpha + \varepsilon \mid \varepsilon < \delta\} > \gamma$$

$$\Downarrow$$

$$\exists\, \varepsilon < \delta \quad \alpha + \varepsilon > \gamma \quad \rightsquigarrow$$

$\blacksquare$

<u>Lemma</u> (divisione euclidea)

$\alpha \neq 0$

Dati $\alpha, \gamma \in \mathrm{Ord}$ esistono unici $\beta, \rho \in \mathrm{Ord}$ tali che $\alpha \cdot \beta + \rho = \gamma$ e $\rho < \alpha$

<u>Dim</u>

<u>Unicità</u> Basta dire che β è unico.

Per assurdo supponiamo

$\rho, \rho' < \alpha$

$$\alpha \cdot \beta + \rho = \gamma = \alpha \cdot \beta' + \rho'$$

sono $\beta < \beta'$

$s(\beta) \leq \beta'$

$$\underline{\alpha \cdot \beta + \rho} < \alpha \cdot \beta + \alpha = \alpha \cdot s(\beta) \leq \alpha \cdot \beta' \leq \underline{\alpha \cdot \beta' + \rho'}$$

$\rightsquigarrow$

Osservazione $\qquad \alpha \cdot \beta + \rho = \gamma \qquad \rho < \alpha$

Sia δ minimo t.c. $\alpha \cdot \delta > \gamma$

Caso $\delta = s(\beta) \Rightarrow \alpha \cdot \beta \leq \gamma$

Per il lemma precedente esiste ρ t.c.

$$\alpha \cdot \beta + \rho = \gamma$$

Vogliamo dire che $\rho < \alpha$. Per assurdo $\alpha \leq \rho$

$$\alpha \cdot \beta + \rho \geq \alpha \cdot \beta + \alpha = \alpha \cdot s(\beta) = \alpha \cdot \delta > \gamma \quad \lightning$$

Se per assurdo δ è limite

$$\alpha \cdot \delta > \gamma \Rightarrow \sup \{ \alpha \cdot \varepsilon \mid \varepsilon < \delta \} > \gamma$$

$$\Downarrow$$

$$\exists \varepsilon < \delta \quad \alpha \cdot \varepsilon > \gamma \quad \lightning$$

∎

Teo (Forma normale di Cantor)

Ogni $\alpha \in Ord$ si scrive in maniera unica come somma finita

$$\alpha = \omega^{\beta_1} k_1 + \omega^{\beta_2} \cdot k_2 + \dots + \omega^{\beta_n} \cdot k_n$$

con $\beta_1 > \beta_2 > \dots > \beta_n$ e $k_1, k_2 \dots k_n \in \omega \setminus \{0\}$.

$$\omega^{\omega^{\omega^2 \cdot 2 + \omega^0 \cdot 1} + \omega^{\omega^4 \cdot 7} + \omega^{15} \cdot 19} + \omega^{\omega^5 \cdot 9} + \omega^0 \cdot 4$$

$\alpha = \omega^{\alpha} \cdot 1$

Dim

Esistenza $\alpha = \omega^{\beta_1} k_1 + \omega^{\beta_2} \cdot k_2 + \dots + \omega^{\beta_n} \cdot k_n$

Sia γ minimo t.c. $\omega^{\gamma} > \alpha$ *

Caso $\gamma = s(\beta_1) \Rightarrow \omega^{\beta_1} \leq \alpha$

Per la divisione euclidea esistono δ, ρ per ω^{β_1}

$$\alpha = \omega^{\beta_1} \cdot \delta + \rho$$

• $\delta \in \omega$ Per assurdo supponiamo $\omega \leq \delta$

$$\omega^{\gamma} = \omega^{s(\beta_1)} = \omega^{\beta_1} \cdot \omega \leq \omega^{\beta_1} \cdot \delta \leq \omega^{\beta_1} \cdot \delta + \rho = \alpha$$ ↯

* Esistenza di γ. $\omega^{s(\alpha)} > \alpha$

$\xi \mapsto \omega^{\xi}$ è crescente strettamente

$\Downarrow$

$\forall \xi \quad \omega^{\xi} \geq \xi$

$\Downarrow$

$\omega^{s(\alpha)} \geq s(\alpha) > \alpha$

Poniamo $k_1 \overset{\text{def.}}{=} \delta$.

Dobbiamo dimostrare che $\rho < \alpha$

$$\alpha = \omega^{\beta_1} \cdot \delta + \rho \Rightarrow \alpha \geq \omega^{\beta_1} > \rho$$

Per ipotesi induttiva

$$p = w^{\beta_2} \cdot k_2 + \dots + w^{\beta_n} \cdot k_n$$

rappresentazione di Cantor.

$$\Downarrow$$

$$a = w^{\beta_1} \cdot k_1 + w^{\beta_2} k_2 + \dots + w^{\beta_n} \cdot k_n$$

Possiamo dire che $\beta_1 > \beta_2 > \dots > \beta_n$

Per assurdo, assumiamo $\beta_2 \geq \beta_1$

$$a = w^{\beta_1} \cdot k_1 + w^{\beta_2} k_2 + \dots \geq w^{\beta_1} k_1 + w^{\beta_2} \geq w^{\beta_1} \cdot k_1 + w^{\beta_1}$$

$\|$

$w^{\beta_1} \cdot k_1 + \rho$ con $\rho < w^{\beta_1}$

$$\rho < w^{\beta_1} \Rightarrow w^{\beta_1} \cdot k_1 + \rho < w^{\beta_1} \cdot k_1 + w^{\beta_1}$$