Consideriamo, per assurdo, il minimo con due f.n. diverse:

$$\alpha = \underline{w^{\beta_1} k_1} + w^{\beta_2} \cdot k_2 + \dots + w^{\beta_n} \cdot k_n$$

$$\alpha = \underline{w^{\beta_1'} k_1'} + w^{\beta_2'} k_2' + \dots + w^{\beta_n'} k_n'$$

Basta dimostrare $w^{\beta_1} k_1 = w^{\beta_1'} k_1'$

$$\alpha = w^{\beta_1} k_1 + \underline{w^{\beta_2} \cdot k_2 + \dots + w^{\beta_n} \cdot k_n} \quad \leftarrow \gamma$$

$$\alpha = w^{\beta_1} k_1 + \underline{w^{\beta_2'} k_2' + \dots + w^{\beta_n'} k_n'}$$

$$\gamma < w^{\beta_2} (k_2 + \dots + k_n) < w^{\beta_2} \cdot w = w^{s(\beta_2)} \leq w^{\beta_1} \leq \alpha$$

$$\alpha = \underline{w^{\beta_1} k_1} + \dots$$

$$\alpha = \underline{w^{\beta_1'} k_1'} + \dots$$

Dimostriamo che $\beta_1 = \beta_1'$ per assurdo $\beta_1 < \beta_1'$

$$\alpha = w^{\beta_1} k_1 + \dots < w^{s(\beta_1)} \leq w^{\beta_1'} \leq \alpha$$

$$\alpha = \underline{w^{\beta_1} k_1} + \dots$$

$$\alpha = \underline{w^{\beta_1} k_1'} + \dots \qquad k_1 < k_1'$$

$$\omega^{\alpha_1} k_1 + \ldots = \omega^{\alpha_1}(k_1 + \Delta) + \ldots$$

$$\ldots = \omega^{\alpha_1}\Delta + \ldots$$

$$\wedge$$

$$\omega^{\alpha_1}$$

Def Sia $F : Ord \to Ord$ una f. classe crescente, Se $\forall \lambda$ limite $F(\lambda) = \sup F[\lambda]$ allora F si dice continua.

Prop Sia $F : Ord \to Ord$. crescente e continua, e sia $\alpha \in Ord$, allora esiste $\beta \geq \alpha$ tale che $F(\beta) = \beta$. "β è un punto fisso di F".

Dim Definiamo per ricorrenza

$$\beta_0 = \alpha \qquad \beta_{n+1} = F(\beta_n) \quad \text{con } n \in \omega$$

Poniamo $\beta = \sup\{\beta_n \mid n \in \omega\}$

- $\forall n \ \beta_n \geq \alpha$ facile induzione

  $\Rightarrow \beta \geq \alpha$.

- β è limite $\quad \beta_{S(n)} = F(\beta_n) > \beta_n$

- $F(\sup\{\beta_n \mid n < \omega\}) = \sup\{F(\beta_n) \mid n < \omega\}$

  $F(\beta)$ $\qquad\qquad$ β_{n+1} $\qquad = \beta$

Def. ξ_0 è il minimo t.c. $\omega^{\xi_0} = \xi_0$

ξ_α = il minimo ordinale tale $\omega^{\xi_\alpha} = \xi_\alpha$ e

$\forall \beta < \alpha \quad \xi_\beta < \xi_\alpha$.

—

Lemma Siano α, β, γ ordinali. Tali che

$$\alpha < \omega^\beta \leq \gamma$$

Allora $\alpha + \gamma = \gamma$.

Dim

Idea: ridursi al caso particolare $n + \omega = \omega$ ($< \omega$)

Voglio dimostrare $\alpha + \gamma \leq \gamma$ — $\boxed{\alpha < \omega^\beta \leq \gamma}$

$$\alpha = \omega^{\delta_1} \cdot k_1 + \dots + \omega^{\delta_n} k_n \leq \omega^{\delta_1} \underbrace{(k_1 + \dots + k_n)}_{k}$$

$$\omega^{\delta_1} \leq \alpha \leq \omega^{\delta_1} \cdot k$$

Osservo che $\omega^{\delta_1} < \omega^\beta$, quindi $\omega^{s(\delta_1)} \leq \omega^\beta \leq \gamma$.

Scrivo $\gamma = \omega^{s(\delta_1)} + \gamma'$

$$\alpha + \gamma \leq \omega^{\delta_1} k + \omega^{s(\delta_1)} + \gamma' \qquad \gamma = \omega^{\delta_1} \cdot \omega + \gamma'$$

$$= \omega^{\delta_1} k + \underbrace{\omega^{\delta_1} \cdot \omega}_{\omega} + \gamma' = \omega^{\delta_1} (k + \omega) + \gamma'$$

∎

Operazioni in Somma normale di Cantor

$w^{\alpha_1} k_1 < w^{\alpha_2} \leq w^{\alpha_2} k_2$

$$w^{\alpha_1} k_1 + w^{\alpha_2} k_2 = \begin{cases} w^{\alpha_2} k_2 & \alpha_1 < \alpha_2 \\ w^{\alpha_1}(k_1 + k_2) & \alpha_1 = \alpha_2 \\ w^{\alpha_1} k_1 + w^{\alpha_2} k_2 & \alpha_1 > \alpha_2 \end{cases}$$

$(\quad) \cdot w^{\alpha} \cdot k$

- $\left(w^{\alpha_1} k_1 + w^{\alpha_2} k_2 + \dots + w^{\alpha_n} k_n \right) \cdot k$

$= w^{\alpha_1} k_1 \cdot k + w^{\alpha_2} k_2 + \dots + w^{\alpha_n} k_n$

$\left(w^2 \cdot 2 + w^4 \cdot 9 + w \right) \cdot 2 =$

$= w^2 \cdot 2 + w^4 \cdot 9 + w + w^2 \cdot 2 + w^4 \cdot 9 + w$

$= w^2 \cdot 4 + w^4 \cdot 9 + w$

$\alpha > 0$

- $\left(w^{\alpha_1} k_1 + w^{\alpha_2} k_2 + \dots + w^{\alpha_n} k_n \right) \cdot w^{\alpha} = w^{\alpha_1 + \alpha}$

$w^{\alpha_1 + \alpha} \leq \underbrace{\left(w^{\alpha_1} k_1 + w^{\alpha_2} k_2 + \dots + w^{\alpha_n} k_n \right)}_{\geq w^{\alpha_1}} \cdot w^{\alpha}$

Occorre dimostrare

$$\underbrace{\left(w^{\alpha_1} k_1 + w^{\alpha_2} k_2 + \dots + w^{\alpha_n} k_n \right)}_{\leq w^{\alpha_1} k} \cdot w^{\alpha} \leq w^{\alpha_1 + \alpha}$$

$\alpha > 0 \qquad \alpha = 1 + \beta$

$$w^{\alpha_1} k \, w^{\alpha} = w^{\alpha_1} w^{\alpha}$$

$$w^{\alpha_1} k \, w^{1+\beta} = w^{\alpha_1} \underbrace{k w} \, w^{\beta} = w^{\alpha_1} \cdot w \, w^{\beta} = w^{\alpha_1} w^{1+\beta}$$

$$= w^{\alpha_1} \cdot w^{\alpha}$$

$$\left(\qquad \right)^{w^{\alpha}} \longrightarrow k^{w^{1+\alpha}} = w^{w^{\alpha}}$$

$$\longrightarrow \underbrace{\left(w^{\alpha_1} k_1 + \dots \right)^{w^{\alpha}} = w^{\alpha_1 w^{\alpha}}}$$

$\uparrow$

La solita disuguaglianza.

$$k^{w^{1+\alpha}} = w^{w^{\alpha}}$$

$$\| \qquad \|$$

$$k^{w \cdot w^{\alpha}} = \left(k^{w} \right)^{w^{\alpha}}$$

$\square$

- $(w+1)^2 = (w+1)(w+1) = \underbrace{(w+1)}_{w^2} w + (w+1) = w^2 + w + 1$
- $(w+1)^2 \cdot n$
- $(w+1)^2 \cdot w$
- $(w+1)^3 = (w+1)^2 (w+1) = (w^2 + w + 1)(w+1)$
- $(w+1)^n$ $= (w^2 + w + 1) w + (w^2 + w + 1)$

$\overset{?}{=} w^n + w^{n-1} + \dots + w + 1$ $= w^3 + w^2 + w + 1$

Per induzione

$$(w+1)^{n+1} = (w+1)^n (w+1) = (w^n + \dots + 1)(w+1)$$

$$(\text{——}) w + (\text{——}) \cdot 1$$

$$w^{n+1} + w^n + \dots + 1.$$

- $(w+1)^w = w^w$

esame del 27.1.2020

- $(2 \cdot w^2 + w \cdot 3 + 7)^3$

$(w^2 + w \cdot 3 + 7)^2 =$

$= (w^2 + w \cdot 3 + 7) \cdot w^2 + (w^2 + w \cdot 3 + 7) \cdot w \cdot 3 + (w^2 + w \cdot 3 + 7) \cdot 7$

$= w^4$ $w^3 \cdot 3$ $w^2 \cdot 7 + w \cdot 3 + 7$

$= \underbrace{w^4 + w^3 \cdot 3 + w^2 \cdot 7 + w \cdot 3 + 7}_{\square}$

$(w^2 + w \cdot 3 + 7)^3 = \square\, w^2 + \square\, w \cdot 3 + \square \cdot 7$

w^6 $w^5 \cdot 3 + w^4 \cdot 7 +$ $w^3 \cdot 3 + w^2 \cdot 7 + w \cdot 3 + 7$

$= w^6 + w^5 \cdot 3 + w^4 \cdot 7 + w^3 \cdot 3 + w^2 \cdot 7 + w \cdot 3 + 7$

Gli aleff.

A è numerabile

$$\begin{array}{c} \equiv \\ |A| = |\omega| \\ \equiv \\ |A| = \aleph_0 \end{array}$$

ω_0

$\omega_1 \overset{\text{def.}}{=}$ il più piccolo tale che $|\omega_1| > |\omega_0|$

$\vdots$

Oss. Se $\alpha, \beta \in \mathrm{Ord}$

- $|\alpha| < |\beta|$
- $|\alpha| = |\beta|$
- $|\alpha| > |\beta|$