Teo (Tarski)

$$\left(\forall X \text{ infinito} \to |X \times X| = |X| \right) \Rightarrow AC$$

Dim

Sia A infinito, vogliamo bene ordinare A.

Uso l'equazione per $X = \underbrace{A \sqcup H(A)}$

$$|A \times H(A)| \leq |X \times X| = |X| = |A \sqcup H(A)|$$

Quindi esiste f iniettiva

$$f : A \times H(A) \longrightarrow A \sqcup H(A)$$

$$\cup\!\!\mid$$

$$f\,[\{a\} \times H(A)] \subseteq A \sqcup H(A)$$

$$\subset A \times \{0\} \cup H(A) \times \{1\}$$

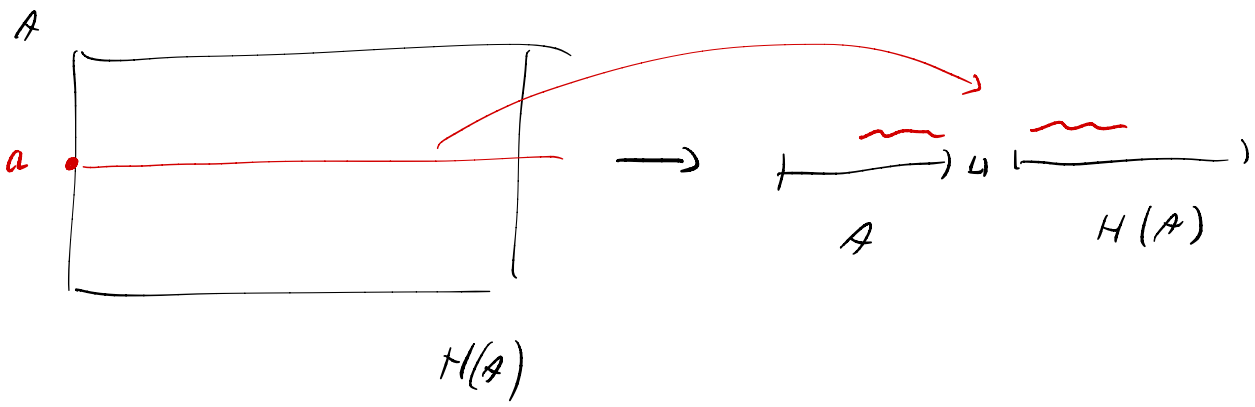
Supponiamo per assurdo che

$$f\,[\{a\} \times H(A)] \subseteq A \leftarrow A \times \{0\}$$

Allora

$$b \quad H(A) \longrightarrow A$$

$$x \longmapsto f(a, x) \quad \text{è in.}$$



Definisco

$$g(a) = \min \{ \alpha \in H(A) \mid \overset{(\alpha,1)}{\alpha} \in f[\{a\} \times H(A)] \}$$

$$g : A \to H(A) \text{ in.}$$

Aritmetica cardinale

$f : I \to \text{Ord. iniziali}$

Sia $\{k_i\}_{i \in I}$ una successione di cardinali con I un insieme.

$$\sum_{i \in I} k_i = \left| \bigcup \{ k_i \times \{i\} \mid i \in I \} \right|$$

$$\prod_{i \in I} k_i = \left| \{ f \in {}^I(\sup_{i \in I} k_i) \mid \forall i \in I \ f(i) \in k_i \} \right|$$

Oss. Se $\{A_i\}_{i \in I}$ sono insiemi a due a due disgiunti, allora $\left| \bigcup \{ A_i \mid i \in I \} \right| = \sum_{i \in I} \underbrace{|A_i|}_{f_i : k_i \to A_i \ \text{bi}}$

Oss. In generale

$$\left| \bigcup \{ A_i \mid i \in I \} \right| \leq \sum_{i \in I} |A_i|$$

Oss. $$\left| \{ f \in {}^I\left(\bigcup_{i \in I} A_i\right) \mid \forall i \; f(i) \in A_i \} \right| = \prod_{i \in I} |A_i|$$

Oss. Se $\forall i \in I \quad k_i \leq \lambda_i$

$$\sum_{i \in I} k_i \leq \sum_{i \in I} \lambda_i \qquad \prod_{i \in I} k_i \leq \prod_{i \in I} \lambda_i$$

$$k_i < \lambda_i \qquad \sum_{i \in I} k_i < \prod_{i \in I} \lambda_i$$

$$k < 2^k$$

$$1 < 2 \qquad \underbrace{\sum_{i \in k} 1}_{= k} < \underbrace{\prod_{i \in k} 2}_{= 2^k}$$

Oss. Sia $I = I_1 \cup I_2$ con $I_1 \cap I_2 = \emptyset$

$$\sum_{i \in I} k_i = \left(\sum_{i \in I_1} k_i \right) + \left(\sum_{i \in I_2} k_i \right)$$

$$\prod_{i \in I} k_i = \left(\prod_{i \in I_1} k_i \right) \cdot \left(\prod_{i \in I_2} k_i \right)$$

$$\sum_{i\in I} k = |I|\cdot k \qquad \prod_{i\in I} k = k^{|I|}$$

$$\left(\prod_{i\in I} k_i\right)^{\lambda} = \prod_{i\in I} k_i^{\lambda}$$

Prop pr. k_i card. infinis :

$$\sum_{i\in I} k_i = \max\left(|I|, \sup_{i\in I} k_i\right)$$

Dem

(≥) $\sum_{i\in I} k_i \geq k_i$ $\qquad \sum_{i\in I} k_i \geq |I|$

(α, i) $\qquad (0, i)$

$\alpha \in k_i$

$$\sum_{i\in I} k_i \geq \max\left(|I|, \sup_{i\in I} k_i\right)$$

(≤) $$\sum_{i\in I} k_i = \left|\left\{(\alpha, i) \in \left(\sup_{i\in I} k_i\right) \times I \mid \alpha \in k_i\right\}\right|$$

$$= \left|\bigcup \{k_i \times \{i\} \mid i \in I\}\right|$$

$$\left[\sup_{i\in I} k_i\right) \times I$$

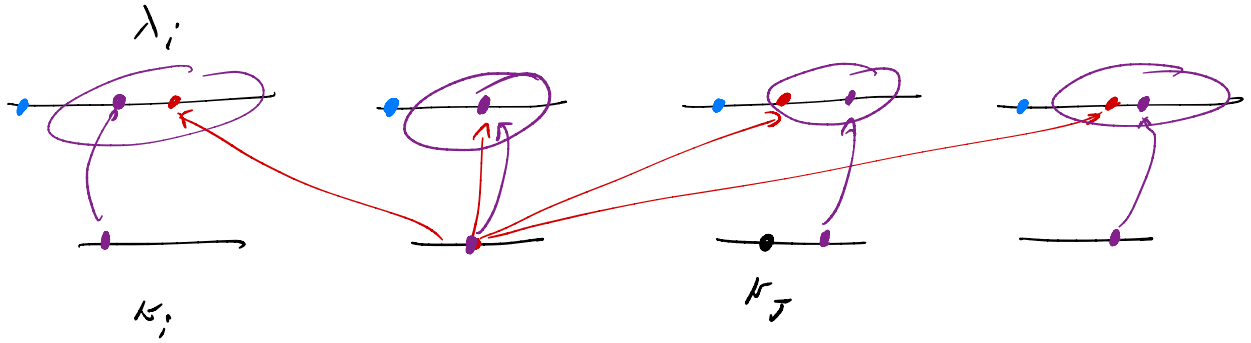
$$\left| \quad \downarrow \quad \right| = \left(\sup k_i\right)\cdot |I| = \max(\dots)$$

□

Tm $\forall i \in I \quad k_i < \lambda_i \Rightarrow \sum_{i\in I} k_i < \prod_{i\in I} \lambda_i$

Dim Supponiamo per assurdo $\sum_{i\in I} k_i \geq \prod_{i\in I} \lambda_i$

"esiste f suriettiva: $\sum k_i \to \prod \lambda_i$"



$|K_i| = k_i$ $\quad K_i$ disgiunti

$|\Lambda_i| = \lambda_i$ $\quad \Lambda = \bigcup \Lambda_i$

Supponiamo per assurdo

$$f : \bigcup K_i \to \underbrace{\{ f \in {}^{I}\Lambda \mid \forall i \ f(i) \in \Lambda_i \}}_{Y}$$

suriettiva.

$$f_i : K_i \to \Lambda_i$$

$$a \longmapsto f(a)(i)$$

Siccome $|K_i| = k_i < \lambda_i = |\Lambda_i|$, f_i non è sur.

Sia $x_i \in \Lambda_i \setminus \mathrm{Im}(f_i)$

La funzione $b : i \mapsto x_i$ $\quad b : I \to \Lambda$

appartiene a Y. Per l'ipotesi assurda esiste

$a \in \bigcup K_i$ tale che $b = f(a)$ Sia $a \in K_i$

$$f_i(a) = f(a)(i) = x_i$$

quindi $x_i \in \operatorname{Im} f_i$ $\blacksquare$

<u>Con</u> $\quad \nu < 2^{\nu}$

<u>Con</u> $\quad 2^{\aleph_0} \neq \aleph_\omega$

<u>Dim</u>

$$\aleph_n < \aleph_\omega$$
$$\Downarrow$$

$$\aleph_\omega = \max\Big(\underbrace{\sup_{n\in\omega} \aleph_n}_{\aleph_\omega}, \aleph_0\Big) = \sum_{n\in\omega} \aleph_n < \prod_{n\in\omega} \aleph_\omega = \aleph_\omega^{\aleph_0}$$

$$\aleph_\omega < \aleph_\omega^{\aleph_0}$$

Però $\quad \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$

$\blacksquare$

Def (cofinalità – v. 1)

$\mathrm{cof}(\kappa)$ = il minimo cardinale λ tale che esistono $\kappa_i < \kappa$ per $i \in \lambda$ con

$$\sum_{i<\lambda} \kappa_i = \kappa$$

Es. $\mathrm{cof}(\aleph_\omega) = \aleph_0$ ← singolare

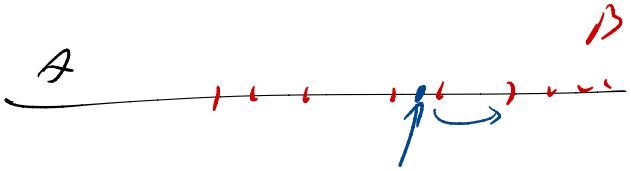
$\mathrm{cof}(\aleph_{\omega_2}) = \aleph_{\omega_2}$ ← regolare

$\left[\sum_{i<\lambda} \kappa_i \leq \sum_{i<\lambda} \aleph_{\omega_1} = \max(\lambda, \aleph_{\omega_1}) \right.$

Def (cofinalità – v. 2)

Sia $(A, <)$ un insieme tot. ordinato.

$B \subseteq A$ è cofinale in A se $\forall a \in A \; \exists b \in B \; a \leq b$



$\mathrm{cof}(A, <)$ = minimo cardinalità di un $B \subseteq A$ cofinale in A.

$\kappa^{\mathrm{cof}\,\kappa} > \kappa$ $\qquad$ $\mathrm{cof}(\kappa^\lambda) > \lambda$