Def (cofinalità v. 1)

Sia k un cardinale infinito

$\mathrm{cof}(k) \overset{\text{def.}}{=}$ la minima cardinalità di una famiglia di cardinali $< k$ la cui somma è k.

Def. (cofinalità v. 2)

Sia $(A, <)$ tot. ordinato

$\mathrm{cof}(A, <) \overset{\text{def}}{=}$ la minima cardinalità di un $B \subseteq A$ cofinale

$\forall a \in A \ \exists b \in B \ a \leq b$

CONV ORDINALE!

Prop. Se k è un card. infinito $\mathrm{cof}_{v.1}(k) = \mathrm{cof}_{v.2}(k)$

Dim

$\mathrm{cof}_{v_1} \leq \mathrm{cof}_{v_2}$

Sia A cofinale in k con $|A| = \mathrm{cof}_{v_2}(k)$

$\sup A = k \rightarrow k = \bigcup A$

$$|k| \leq \sum_{a \in A} \underbrace{|a|}_{< k} = \max\left(|A|, \sup_{a \in A} |a|\right) \leq |k|$$

$\mathrm{cof}_{v_2} \leq \mathrm{cof}_{v_1}$

Sia $k = \sum_{i \in I} k_i$ con $|I| = \mathrm{cof}_{v_1}(k)$

o $|I| = k \longrightarrow$ la tesi è ovvia.

- $|I| < k$ $\quad k = \sum_{i \in I} k_i = \max\left(\sup_{i \in I} k_i, |I|\right)$

$k = \sup_{i \in I} k_i$ ∎

Oss. Dato $(A, <)$, sia B cofinale in A e C cofinale in B, allora C è cofinale in A.

Oss. k cardinale $\longrightarrow$ $\operatorname{cof} k \leq k$

Oss. $\operatorname{cof}(\operatorname{cof}(A, <)) = \operatorname{cof}(A, <)$

Prop. ($\leq$) immediata

($\geq$) Sia B cofinale in A, sia $k = \operatorname{cof}(|B|)$ (di cardinale minimo)

Cerchiamo un insieme $C \subseteq A$ cofinale in A con $|C| = k$.

Due casi:

- se $k = |B|$ prendo $C = B$.
- sempre $k < |B|$

Scrivo $B = \bigcup_{i \in I} B_i$ con $|B_i| < |B|$ con $|I| = \operatorname{cof}(|B|)$

Per ogni $i \in I$ sia $c_i \in A$ con $\forall b \in B_i \; b < c_i$

Definisco $C = \{c_i \mid i \in I\}$

$|C| \leq |I| = \operatorname{cof}(|B|)$

$\forall a \in A \quad \exists b \in B \quad a \leq b$

$\rightarrow \exists i \; b \in B_i \rightarrow a \leq b < c_i \in C$ ∎

Corollary (del tao. di König) κ inf. $\lambda \geq 2$

$$\kappa < \kappa^{\mathrm{cof}(\kappa)} \qquad \kappa < \mathrm{cof}(\lambda^\kappa)$$

Dim

(1) Sia $\kappa = \sum_{i < \mathrm{cod}(\kappa)} \kappa_i$ con $\kappa_i < \kappa$

$$\kappa = \sum_{i < \mathrm{cod}(\kappa)} \kappa_i < \prod_{i < \mathrm{cod}(\kappa)} \kappa = \kappa^{\mathrm{cof}(\kappa)}$$

(2)

$$\lambda^\kappa < \left(\lambda^\kappa\right)^{\mathrm{cof}(\lambda^\kappa)} = \lambda^{\kappa \cdot \mathrm{cof}(\lambda^\kappa)}$$

$$\kappa < \kappa \cdot \mathrm{cof}(\lambda^\kappa) = \max\left(\kappa, \mathrm{cof}(\lambda^\kappa)\right)$$

$$\kappa < \mathrm{cof}(\lambda^\kappa)$$

Def $\partial(\kappa) \overset{\text{def}}{=} \kappa^{\mathrm{cof}(\kappa)}$

$\kappa = \aleph_\alpha \qquad \kappa^+ = \aleph_{s(\alpha)}$

Teo (Hausdorff)

$$\left(\kappa^+\right)^\lambda = \kappa^\lambda \cdot \kappa^+$$

Es $\aleph_{n+3}^{\aleph_0} = \aleph_n^{\aleph_0} \cdot \aleph_{n+3}$

$$\aleph_{n+3}^{\aleph_0} = \aleph_{n+2}^{\aleph_0} \cdot \aleph_{n+3} = \aleph_{n+1}^{\aleph_0} \cdot \aleph_{n+2} \cdot \aleph_{n+3} =$$

$$= \aleph_n^{\aleph_0} \cdot \aleph_{n+1} \aleph_{n+2} \aleph_{n+3} = \aleph_n^{\aleph_0} \cdot \aleph_{n+3}$$

$\kappa = \aleph_\alpha \to \kappa^+ \overset{\text{def.}}{=} \aleph_{s(\alpha)}$

Teo (Hausdorff)

$$(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$$

Dim

(≥) Immediata

(≤) Se $\kappa^+ \leq \lambda$

$$2^\lambda \leq (\kappa^+)^\lambda \leq \lambda^\lambda = 2^\lambda$$

$$2^\lambda \leq \kappa^\lambda \leq \lambda^\lambda = 2^\lambda$$

$$\hookrightarrow \kappa^\lambda \cdot \kappa^+ = 2^\lambda \cdot \kappa^+ = 2^\lambda$$

Assumiamo $\lambda \leq \kappa$

Lemma $\operatorname{cof}(\kappa^+) = \kappa^+$

$$(\kappa^+)^\lambda = |\{f : \lambda \to \kappa^+\}|$$

Dato $f : \lambda \to \kappa^+$ $\quad |\operatorname{Im} f| \leq \lambda \leq \kappa < \kappa^+$

Quindi $\exists\, \alpha \in \kappa^+$ $\quad \operatorname{Im} f \subseteq \alpha$

$$(\kappa^+)^\lambda = |\{f : \lambda \to \kappa^+\}|$$

$$= \left| \bigcup_{\alpha \in \kappa^+} \{f : \lambda \to \alpha\} \right|$$

$$(\kappa^+)^\lambda \leq \sum_{\alpha < \kappa^+} |\{f : \lambda \to \alpha\}|$$

$$= \sum_{\alpha < \kappa^+} |\alpha|^\lambda \leq \sum_{\alpha < \kappa^+} \kappa^\lambda = \kappa^\lambda \cdot \kappa^+$$

Lemma $\mathrm{cof}(\kappa^+) = \kappa^+$

Dim.

Suponiamo $\lambda = \mathrm{cof}(\kappa^+) < \kappa^+$

$$\kappa^+ = \sum_{i<\lambda} \kappa_i = \max\left(\lambda, \sup_{i<\lambda} \kappa_i\right) \leq \kappa$$

Def. κ cardinale infinito è

- regolare se $\kappa = \mathrm{cof}(\kappa)$
- singolare se $\kappa > \mathrm{cof}(\kappa)$

$\kappa = (\aleph_\alpha)^+$

- successore se $\kappa = \aleph_{s(\alpha)}$ per qualche α
- limite se $\kappa = \aleph_\lambda$ per λ limite
- debolmente inaccessibile se regolare e limite
- limite forte se $\aleph_0 < \kappa$ e $\forall \lambda < \kappa \quad 2^\lambda < \kappa$

• inaccessibile se regolare e limite forte.

Oss λ limite $\rightarrow$ $\mathrm{cof}(\aleph_\lambda) = \mathrm{cof}(\lambda)$

Dim Sia $A \subseteq \lambda$ cofinale in λ

$$\aleph_\lambda = \sup_{\alpha \in \lambda} \aleph_\alpha = \sup_{\alpha \in A} \aleph_\alpha = \sum_{\alpha \in A} \aleph_\alpha$$

$$\rightarrow \quad \mathrm{cof}(\aleph_\lambda) \leq \mathrm{cof}(\lambda)$$

Dunque dim $\mathrm{cof}(\lambda) \leq \mathrm{cof}(\aleph_\lambda)$

Sia A cofinale in ω_λ

Per $a \in A$ $\quad |a| < \aleph_\lambda \qquad |a|^+ < \aleph_\lambda$

$$|a| \leq a < \underbrace{|a|^+}_{= \aleph_{f(a)}} < \aleph_\lambda$$

$\square$

$$\mathrm{cof}(\aleph_\omega) = \mathrm{cof}(\omega) = \aleph_0$$

$$\aleph_\omega < \aleph_\omega^{\aleph_0} \rightarrow \aleph_\omega \neq 2^{\aleph_0}$$

Es

$$\aleph_\omega^{\aleph_1} = \exists(\aleph_1) \cdot \exists(\aleph_\omega)$$