$\{\{\{\{\ldots\}\}\}\}$ $\quad x = \{x\}$ $\quad a = \{b\}$ $\quad b = \{a\}$



$x_1 \ni x_2 \ni x_3 \ni x_4 \ldots$

Def

$$V_0 = \emptyset$$

$$V_{S(\alpha)} = \mathcal{P}(V_\alpha)$$

$$V_\lambda = \bigcup \{ V_\alpha \mid \alpha < \lambda \}$$

$$x \in V_* \overset{(a.)}{\equiv} \exists \alpha \quad x \in V_\alpha$$

"$V_* = \bigcup_{\alpha \in Ord} V_\alpha$" $\qquad \forall x \in V_\alpha \quad x \subseteq V_\alpha$

Lem $\forall \alpha \in Ord \quad V_\alpha$ è transitivo

Dim

$V_{S(\alpha)} = \mathcal{P}(V_\alpha)$ $\qquad$ $V_{S(\alpha)} = \mathcal{P}(V_\alpha)$

- caso successore $\quad x \in \mathcal{P}(V_\alpha) \to x \subseteq \mathcal{P}(V_\alpha)$

  ip. ind. $\quad y \in V_\alpha \to y \subseteq V_\alpha$

$$x \in \mathcal{P}(V_\alpha) \to x \subseteq V_\alpha \to \forall y \in x \quad y \in V_\alpha$$

$$\downarrow$$

$$\forall y \in x \quad y \subseteq V_\alpha$$

- cojo límite

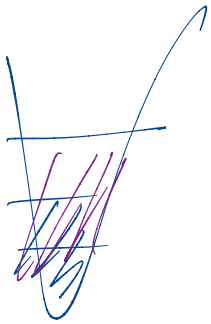
$$x \in \bigcup_{\alpha \in \lambda} V_\alpha \to x \in V_{\bar\alpha} \to x \subseteq V_{\bar\alpha} \to x \subseteq \bigcup_{\alpha\in\lambda} V_\alpha$$

Con/Def

V_* es una clase transitiva

$$\forall x \in V_* \quad x \subseteq V_*$$

Lema $\alpha \le \beta \to V_\alpha \subseteq V_\beta$



Def $\alpha < \beta \to V_\alpha \subseteq V_\beta$

Ind. transfinita sobre β

- sucesor $\quad \alpha < s(\beta) \to V_\alpha \subseteq V_{s(\beta)}$

$$\alpha < \beta \to V_\alpha \subseteq V_\beta$$

$$\alpha < s(\beta) \begin{cases} \alpha = \beta & V_\alpha \in \mathcal{P}(V_\alpha) = V_{s(\beta)} \\ \alpha < \beta & V_\alpha \subseteq V_\beta \end{cases}$$

transf. de V_β

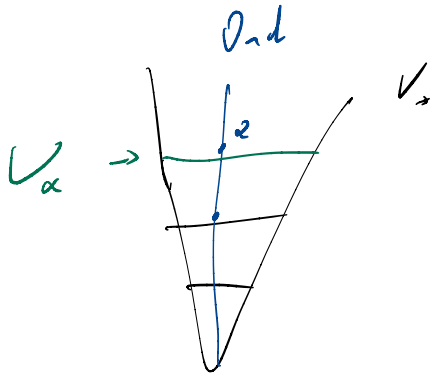
$$\forall x \in V_\alpha \quad x \in V_\beta$$

$$\forall x \in V_\alpha \quad x \subseteq V_\beta$$

$$\forall x \in V_\alpha \quad x \in \mathcal{P}(V_\beta) = V_{s(\beta)}$$

Lemma $\forall \alpha \in Ord \quad \alpha \subseteq V_\alpha \longrightarrow x \in V_{S(x)}$

Dim



- $\emptyset \in \{\emptyset\} = V_{S(\emptyset)}$
- $S(\alpha) \subseteq V_{S(\alpha)}$

  $\alpha \subseteq V_\alpha$

  $\alpha \cup \{\alpha\} \subseteq V_{S(\alpha)}$

  $\in V_{S(\alpha)} \quad \subseteq \mathcal{P}(V_\alpha) = V_{S(\alpha)}$

  $\subseteq V_{S(\alpha)}$
- caso limite: $\lambda \subseteq V_\lambda$

$$\lambda = \bigcup_{\alpha \in \lambda} \alpha \subseteq \bigcup_{\alpha \in \lambda} V_\alpha = V_\lambda$$

□

Def Data una formula φ la formula φ relativizzata a C è quella che si ottiene rimpiazzando $\forall x \rightsquigarrow \forall x \in C$

$\exists x \rightsquigarrow \exists x \in C$

Prop Gli assiomi relativizzati a V_α sono validi

(1) $\exists x \forall y \; y \notin x$

$\exists x \in V_* \quad \forall y \in V_* \quad y \notin x$

Basta tomar $\emptyset \in V_*$

$\emptyset \in V_1 = \{\emptyset\}$

(2) $\forall a \forall b \left(\forall x \quad x \in a \leftrightarrow x \in b \right) \rightarrow a = b$

$\forall a \in V_* \; \forall b \in V_* \left(\forall x \in V_* \quad x \in a \leftrightarrow x \in b \right) \rightarrow a = b$

Suponemos $a \in V_*$ $b \in V_*$ ahora

$\forall x \in a \quad x \in V_* \qquad \forall x \in b \quad x \in V_*$

$\forall x \quad x \notin V_* \rightarrow \left(x \in a \leftrightarrow x \in b \right)$

F F

Asumimos $\forall x \in V_* \quad x \in a \leftrightarrow x \in b$

$\Rightarrow \forall x \quad x \in a \leftrightarrow x \in b \Rightarrow a = b$

(4) Para

$\forall a \; \forall b \; \exists c \; \forall x \quad x \in c \leftrightarrow (x = a \vee x = b)$

$\forall a \in V_* \; \forall b \in V_* \; \exists c \in V_* \; \forall x \in V_* \; x \in c \leftrightarrow (x = a \vee x = b)$

Si $a, b \in V_*$ demostramos que $\{a, b\} \in V_*$

Supponiamo $a \in V_\alpha$ $\quad b \in V_\beta$

SPDG $\alpha \leq \beta$ $\quad a, b \in V_\beta$

$$\{a, b\} \subseteq V_\beta$$

$$\{a, b\} \in V_{S(\beta)}$$

(8) Data $F : V_\alpha \to V_\alpha$ f. classe, data $X \in V_\alpha$

$F[X] \in V_\alpha$

Data $a \in X$ esiste il minimo $\alpha \in Ord$ t.c. $F(a) \in V_\alpha$

sia questo α_a. Considero $\beta = \sup \{\alpha_a \mid a \in X\}$

$\forall a \in X \quad F(a) \in V_\beta \Rightarrow F[X] \subseteq V_\beta$

$\Downarrow$

$F[X] \in V_{S(\beta)}$

■

(9) Data $x \in V_\alpha$ esiste il minimo α t.c. $x \in V_\alpha$

ed è necessariamente successore $\alpha = S(\beta)$.

Def. Il rango di x è β

$$x \in V_\gamma \longleftrightarrow \mathrm{rank}(x) < \gamma$$

Oss $x \in y \in V_\alpha \rightarrow \mathrm{rank}(x) < \mathrm{rank}(y)$

$y \in V_{s(\mathrm{rank}(y))} \rightarrow y \in \mathcal{P}(V_{\mathrm{rank}(y)})$

$\downarrow$

$x \in V_{\mathrm{rank}(y)}$

$\downarrow$

$\mathrm{rank}(x) < \mathrm{rank}(y)$

Cor Non esiste una catena infinita discendente

$x_0 \ni x_1 \ni x_2 \ni \dots$

di elementi di V_α (con $x_0 \in V_\alpha$).

(10) Assioma di buona fondazione

$\forall S \neq \emptyset \quad \exists x \in S \quad x \cap S = \emptyset.$

Cor $\forall X$ esiste il minimo insieme transitivo Y t.c. $X \subseteq Y$.

Dim $Y = X \cup \bigcup X \cup \bigcup\bigcup X$

$Y_0 = X$

$Y_{s(n)} = \bigcup Y_n$ $\qquad$ $Y_\omega = \bigcup_{n \in \omega} Y_n$

$\square$

Def L'insieme Y si chiama **chiusura transitiva** di X

$$Y = cl(X).$$

Prop Principio di $\in$-induzione

$$\left(\forall x \left(\forall y \in x \;\; \varphi(y)\right) \to \varphi(x)\right) \to \forall x \;\; \varphi(x)$$

$$\left(\forall x \;\; x \subseteq C \to x \in C\right) \to C = V$$

Dim Per assurdo, supponiamo $\neg \varphi(x)$

Consideriamo $A = \{ y \in cl(\{x\}) \mid \neg \varphi(y) \}$

Sappiamo $z \in A$ t.c. $z \cap A = \emptyset$

$$\left.\begin{array}{l} \neg \varphi(z) \to \exists t \in z \;\; \neg \varphi(t) \\ z \in cl(\{x\}) \to \forall t \in z \;\; t \in cl(\{x\}) \end{array}\right\} \begin{array}{l} t \in z \\ t \in A \end{array}$$

