Sistema completo (ND — Natural Deduction)

Simboli logici: $\top\ \bot\ \wedge\ \vee\ \to\ \neg\ \forall\ \exists\ =$

$Ax\ \dfrac{}{\varphi \vdash \varphi}$ $\qquad$ $W\!k\ \dfrac{T \vdash \varphi}{T' \vdash \varphi}$ con $T \subseteq T'$

$In_{\top}\ \dfrac{}{\vdash \top}$ $\qquad$ $El_{\bot}\ \dfrac{T \vdash \bot}{T \vdash \varphi}$

$In_{\wedge}\ \dfrac{T \vdash \varphi \quad T \vdash \psi}{T \vdash \varphi \wedge \psi}$ $\qquad$ $El_{\wedge}\ \dfrac{T \vdash \varphi \wedge \psi}{T \vdash \varphi}$ $\qquad$ $El_{\wedge}\ \dfrac{T \vdash \varphi \wedge \psi}{T \vdash \psi}$

$In_{\vee}\ \dfrac{T \vdash \varphi}{T \vdash \varphi \vee \psi}$ $\qquad$ $In_{\vee}\ \dfrac{T \vdash \psi}{T \vdash \varphi \vee \psi}$ $\qquad$ $El_{\vee}\ \dfrac{T \vdash \varphi \vee \psi \quad T, \varphi \vdash \theta \quad T, \psi \vdash \theta}{T \vdash \theta}$

$In_{\to}\ \dfrac{T, \varphi \vdash \psi}{T \vdash \varphi \to \psi}$ $\qquad$ $El_{\to}\ \dfrac{T \vdash \varphi \to \psi \quad T \vdash \varphi}{T \vdash \psi}$

$In_{\neg}\ \dfrac{T, \varphi \vdash \bot}{T \vdash \neg \varphi}$ $\qquad$ $El_{\neg}\ \dfrac{T \vdash \neg\neg\varphi \quad T \vdash \varphi}{T \vdash \psi}$ $\qquad$ $RaA\ \dfrac{T, \neg\varphi \vdash \bot}{T \vdash \varphi}$

$In_{\forall}\ \dfrac{T \vdash \varphi}{T \vdash \forall x_n \varphi}$ con $x_n \notin vl(T)$ $\qquad$ $El_{\forall}\ \dfrac{T \vdash \forall x_n \varphi}{T \vdash \varphi[t/x_n]}$

$In_{\exists}\ \dfrac{T \vdash \varphi[t/x_n]}{T \vdash \exists x_n \varphi}$ $\qquad$ $El_{\exists}\ \dfrac{T \vdash \exists x_n \varphi \quad T, \varphi \vdash \psi}{T \vdash \psi}$ con $x_n \notin vl(T, \psi)$

$In_{=}\ \dfrac{}{\vdash t = t}$ $\qquad$ $El_{=}\ \dfrac{T \vdash s = t \quad T \vdash \varphi[s/x_n]}{T \vdash \varphi[t/x_n]}$

Sistema ridotto ($ND_{\to\perp\exists}$)

Simboli logici: $\perp \to \exists =$

Ax $\dfrac{}{\varphi \vdash \varphi}$ $\quad W_k$ $\dfrac{T \vdash \varphi}{T' \vdash \varphi}$ con $T \subseteq T'$ $\quad RaA$ $\dfrac{T, \varphi \to \perp \vdash \perp}{T \vdash \varphi}$

$I_{n\to}$ $\dfrac{T, \varphi \vdash \psi}{T \vdash \varphi \to \psi}$ $\quad El_{\to}$ $\dfrac{T \vdash \varphi \to \psi \quad T \vdash \varphi}{T \vdash \psi}$

$I_{n\exists}$ $\dfrac{T \vdash \varphi[t/x_k]}{T \vdash \exists x_k \varphi}$ $\quad El_{\exists}$ $\dfrac{T \vdash \exists x_k \varphi \quad T, \varphi \vdash \psi}{T \vdash \psi}$ con $x_n \notin var(T, \psi)$

$I_{n=}$ $\dfrac{}{\vdash t = t}$ $\quad El_{=}$ $\dfrac{T \vdash s = t \quad T \vdash \varphi[s/x_k]}{T \vdash \varphi[t/x_k]}$

Assumendo le seguenti definizioni, il sistema completo si ricava come conseguenza di quello ridotto.

Def ($\neg\ T\ \wedge\ \vee\ \forall$)

$\neg\varphi \overset{def.}{=} \varphi \to \perp$ $\quad T \overset{def}{=} \neg\perp$ $\quad \varphi \wedge \psi \overset{def.}{=} \neg(\varphi \to \neg\psi)$

$\varphi \vee \psi \overset{def}{=} \neg(\neg\varphi \wedge \neg\psi)$ $\quad \forall x_k \varphi \overset{def.}{=} \neg\exists x_k \neg\varphi$

Prop.

Valgono $I_{n\neg}$ $El_{\perp}$ $El_{\neg}$ I_{nT} $I_{n\wedge}$ $El_{\wedge}$ $El_{\wedge}$ $I_{n\vee}$ $I_{n\vee}$ $El_{\vee}$ $I_{n\forall}$ $El_{\forall}$