Łoś (łoś')

$\forall i \in I \quad M_i \models \varphi \implies \prod M_i/U \models \varphi$

N.B. ↗

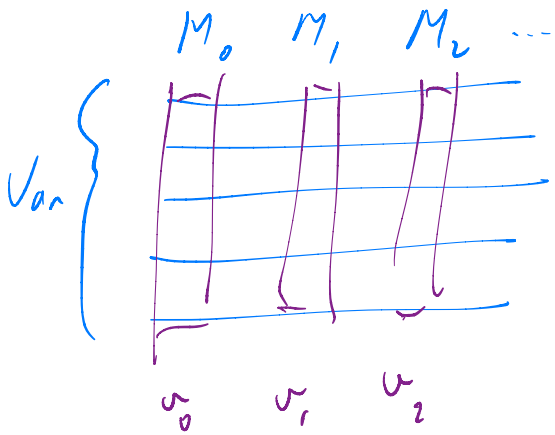
$\prod M_i/U \models \varphi \iff \{ i \mid M_i \models \varphi \} \in U$

⇑

$\prod M_i/U \models [\![t]\!] \varphi \iff \{ i \mid M_i \models [\![v_i]\!] \varphi \} \in U$

$v : Var \to \prod M_i/_{\sim_0}$ $\qquad$ $v_i : Var \to M_i$

$\longmapsto \quad v_i : x_r \to (v(x_r))_r$



$\{ \neg, \wedge \}$
ins. completo di connettivi

$\prod M_i/U \models [\![t]\!] \varphi \iff \{ i \mid M_i \models [\![v_i]\!] \varphi \} \in U$

- φ atomica → esercizio

$\prod M_i/U \models [\![t]\!] \neg \psi$

⇓

$\neg \prod M_i/U \models [\![t]\!] \psi \iff \{ i \mid M_i \models [\![v_i]\!] \psi \} \notin U$

$\prod M_i/U \models [\![u]\!] \neg\varphi$

$\Updownarrow$

$\neg \prod M_i/U \models [\![u]\!]\varphi \iff \{i \mid M_i \models [\![v_i]\!]\varphi\} \notin U$

$\Updownarrow$

$I \setminus \{i \mid M_i \models [\![v_i]\!]\varphi\} \in U$

$\{i \mid \neg M_i \models [\![v_i]\!]\varphi\}$

$\{i \mid M_i \models [\![v_i]\!]\neg\varphi\}$

$\prod M_i/U \models [\![u]\!]\varphi_1 \wedge \varphi_2$

$\Updownarrow$ SOM. DI ŁOŚ

$\prod M_i/U \models [\![u]\!]\varphi_1 \wedge \prod M_i/U \models [\![u]\!]\varphi_2$

$\Updownarrow$ INDUZIONE $\Updownarrow$

$\{i \mid M_i \models [\![v_i]\!]\varphi_1\} \in U \wedge \{i \mid M_i \models [\![v_i]\!]\varphi_2\} \in U$

$\Updownarrow$

$\{i \mid M_i \models [\![v_i]\!]\varphi_1\} \cap \{i \mid M_i \models [\![v_i]\!]\varphi_2\} \in U$

$\{i \mid M_i \models [\![v_i]\!]\varphi_1 \wedge M_i \models [\![v_i]\!]\varphi_2\}$

$\{i \mid M_i \models [\![v_i]\!]\varphi_1 \wedge \varphi_2\}$

$$\prod M_i/U \models \{[v]\}\ \exists x_k \varphi$$

$\Updownarrow$ TARSKI

$$\exists a \in \prod M_i \quad \prod M_i/U \models \{[v[a/x_k]]\}\ \varphi$$

$\Updownarrow$ INDUZIONE

$$\exists a \in \prod M_i \quad \{i \mid M_i \models \{\underbrace{v_i[a_i/x_k]}_{(v[a/x_k])_i}\}\ \varphi\} \in U$$

$P(a_\cdot)$

$\Updownarrow$

$$\exists a \in \prod M_i \quad \{i \mid P(a_\cdot)\} \in U$$

$\Updownarrow$ ovvio.

$$\{i \mid \exists a_i \quad P(a_\cdot)\} \in U$$

$$\{i \mid \exists a_i \quad M_i \models \{v_i[a_i/x_k]\}\ \varphi\} \in U$$

$\Updownarrow$

$$\{i \mid M_i \models \{v_i\}\ \exists x_k\ \varphi\} \in U$$

∎

Teo

Se $T \vDash \varphi$ allora $\exists\, T' \subseteq T$ $|T'| < \aleph_0$ e $T' \vDash \varphi$

Dim (caso $|T| = \aleph_0$)

Sia $T = \{\varphi_0, \varphi_1, \varphi_2, \dots\}$ $\quad T_i := \{\varphi_0, \dots, \varphi_i\}$

Per assurdo $T_i \not\vDash \varphi$

Quindi: $\exists\, M_i$ $\quad M_i \vDash T_i$ e $M_i \vDash \neg\varphi$

Considero $\prod_{i \in \mathbb{N}} M_i / U \vDash \varphi_i$ $\quad \prod M_i / U \vDash \neg\varphi$

Dim (tutti i casi)

F filtro $\mathcal{P}(\mathcal{P}^{fin}(T))$

$X \in F \overset{d.d.}{\Leftrightarrow} \exists\, B \in \mathcal{P}^{fin}(T)\ \{C \in \mathcal{P}^{fin}(T) \mid B \subseteq C\} \subseteq X$

U ultrafiltro su $\mathcal{P}(\mathcal{P}^{fin}(T))$ che estende F.

Per assurdo, dato $A \in \mathcal{P}^{fin}(T)$ esiste M_A tale che

$M_A \vDash A$ $\quad M_A \vDash \neg\varphi$ etc.

verifica dell'assurdo: esercizio.

□

Regole di deduzione

Scopo

$$T \models \varphi \iff T \vdash \varphi$$

Forma: $\dfrac{\text{premesse} \ldots}{\text{conclusione}}$

Sistema completo (ND — Natural Deduction)

Simboli logici: $\top\ \bot\ \wedge\ \vee\ \to\ \neg\ \forall\ \exists\ =$

$Ax\ \dfrac{}{\varphi \vdash \varphi}$ $\qquad$ $W\!k\ \dfrac{T \vdash \varphi}{T' \vdash \varphi}$ con $T \subseteq T'$

$In_{\top}\ \dfrac{}{\vdash \top}$ $\qquad$ $El_{\bot}\ \dfrac{T \vdash \bot}{T \vdash \varphi}$

$In_{\wedge}\ \dfrac{T \vdash \varphi \quad T \vdash \psi}{T \vdash \varphi \wedge \psi}$ $\qquad$ $El_{\wedge}\ \dfrac{T \vdash \varphi \wedge \psi}{T \vdash \varphi}$ $\qquad$ $El_{\wedge}\ \dfrac{T \vdash \varphi \wedge \psi}{T \vdash \psi}$

$In_{\vee}\ \dfrac{T \vdash \varphi}{T \vdash \varphi \vee \psi}$ $\qquad$ $In_{\vee}\ \dfrac{T \vdash \psi}{T \vdash \varphi \vee \psi}$ $\qquad$ $El_{\vee}\ \dfrac{T \vdash \varphi \vee \psi \quad T, \varphi \vdash \theta \quad T, \psi \vdash \theta}{T \vdash \theta}$

$In_{\to}\ \dfrac{T, \varphi \vdash \psi}{T \vdash \varphi \to \psi}$ $\qquad$ $El_{\to}\ \dfrac{T \vdash \varphi \to \psi \quad T \vdash \varphi}{T \vdash \psi}$

$In_{\neg}\ \dfrac{T, \varphi \vdash \bot}{T \vdash \neg \varphi}$ $\qquad$ $El_{\neg}\ \dfrac{T \vdash \neg\neg\varphi \quad T \vdash \varphi}{T \vdash \psi}$ $\qquad$ $RaA\ \dfrac{T, \neg\varphi \vdash \bot}{T \vdash \varphi}$

$In_{\forall}\ \dfrac{T \vdash \varphi}{T \vdash \forall x_n \varphi}$ con $x_n \notin vl(T)$ $\qquad$ $El_{\forall}\ \dfrac{T \vdash \forall x_n \varphi}{T \vdash \varphi[t/x_n]}$

$In_{\exists}\ \dfrac{T \vdash \varphi[t/x_n]}{T \vdash \exists x_n \varphi}$ $\qquad$ $El_{\exists}\ \dfrac{T \vdash \exists x_n \varphi \quad T, \varphi \vdash \psi}{T \vdash \psi}$ con $x_n \notin vl(T, \psi)$

$In_{=}\ \dfrac{}{\vdash t = t}$ $\qquad$ $El_{=}\ \dfrac{T \vdash s = t \quad T \vdash \varphi[s/x_n]}{T \vdash \varphi[t/x_n]}$

Sistema ridotto ($ND_{\to\perp\exists}$)

Simboli logici: $\perp \to \exists =$

Ax $\dfrac{}{\varphi \vdash \varphi}$ $\quad W_k$ $\dfrac{T \vdash \varphi}{T' \vdash \varphi}$ con $T \subseteq T'$ $\quad RaA$ $\dfrac{T, \varphi \to \perp \vdash \perp}{T \vdash \varphi}$

$I_{n\to}$ $\dfrac{T, \varphi \vdash \psi}{T \vdash \varphi \to \psi}$ $\quad El_{\to}$ $\dfrac{T \vdash \varphi \to \psi \quad T \vdash \varphi}{T \vdash \psi}$

$I_{n\exists}$ $\dfrac{T \vdash \varphi[t/x_k]}{T \vdash \exists x_k \varphi}$ $\quad El_{\exists}$ $\dfrac{T \vdash \exists x_k \varphi \quad T, \varphi \vdash \psi}{T \vdash \psi}$ con $x_n \notin var(T, \psi)$

$I_{n=}$ $\dfrac{}{\vdash t = t}$ $\quad El_{=}$ $\dfrac{T \vdash s = t \quad T \vdash \varphi[s/x_k]}{T \vdash \varphi[t/x_k]}$

Assumendo le seguenti definizioni, il sistema completo si ricava come conseguenza di quello ridotto.

Def ($\neg\ T\ \wedge\ \vee\ \forall$)

$\neg\varphi \overset{def.}{=} \varphi \to \perp$ $\quad T \overset{def}{=} \neg\perp$ $\quad \varphi \wedge \psi \overset{def.}{=} \neg(\varphi \to \neg\psi)$

$\varphi \vee \psi \overset{def}{=} \neg(\neg\varphi \wedge \neg\psi)$ $\quad \forall x_k \varphi \overset{def.}{=} \neg\exists x_k \neg\varphi$

Prop.

Valgono $I_{n\neg}$ $El_{\perp}$ $El_{\neg}$ I_{nT} $I_{n\wedge}$ $El_{\wedge}$ $El_{\wedge}$ $I_{n\vee}$ $I_{n\vee}$ $El_{\vee}$ $I_{n\forall}$ $El_{\forall}$