Computabilità

Def (Funzioni primitive ricorsive)

La classe delle f. P.R. è costituita da funzioni $\mathbb{N}^n \to \mathbb{N}$ al variare di $n \in \mathbb{N}$.

- Sono P.R.

$$0 : \mathbb{N} \to \mathbb{N} \qquad S : \mathbb{N} \to \mathbb{N}$$
$$x \mapsto 0 \qquad x \mapsto x+1$$

$$\pi_i^n : \mathbb{N}^n \to \mathbb{N}$$
$$(x_1, \dots x_n) \mapsto x_i$$

- Se $f : \mathbb{N}^k \to \mathbb{N}$, $g_i : \mathbb{N}^n \to \mathbb{N}$ $i = 1 \dots k$ P.R.

allora $\mathbb{N}^n \to \mathbb{N}$

$$(x_1, \dots x_n) \mapsto f(g_1(x_1, \dots x_n), g_2(x_1, \dots x_n) \dots)$$

- Se $h : \mathbb{N}^k \to \mathbb{N}$ P.R.

$g : \mathbb{N}^{k+2} \to \mathbb{N}$ P.R.

Allora l'unica f tale che

$$f(x_1, \dots x_n, 0) = h(x_1, \dots x_n)$$
$$f(x_1, \dots x_n, y+1) = g(x_1, \dots x_n, y, f(x_1, \dots x_n, y))$$

è P.R.

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- Funzioni costanti:

$$f : \mathbb{N}^3 \to \mathbb{N}$$
$$(x,y,z) \mapsto 4$$

$$f(x,y,z) = S(S(S(S(0(\pi_1^3(x,y,z))))))$$

- $x+y \qquad x \cdot y \qquad x^y$

$$f(x_1, \dots x_n, 0) = h(x_1, \dots x_n)$$
$$f(x_1, \dots x_n, y+1) = g(x_1, \dots x_n, y, f(x_1, \dots x_n, y))$$

$$f(x,y) = x+y$$

$$f(x,0) = \pi_1^1(x)$$

$$f(x, y+1) = S(f(x,y))$$

$$g(x,y,z) = S(z)$$
$$= S(\pi_3^3(x,y,z))$$

$x \cdot y$

$$x \cdot 0 = 0$$
$$x \cdot (y+1) = x + x \cdot y$$

- $pred(x) = \begin{cases} 0 & x = 0 \\ x-1 & x \geq 1 \end{cases}$

$pred(0) = 0$

$pred(y+1) = y$

$g(y, pred(y))$ — π_1^2

$f(x_1 \dots x_n, 0) = h(x_1 \dots x_n)$

$f(x_1 \dots x_n, y+1) = g(x_1 \dots x_n, y, f(x_1 \dots x_n, y))$

- $x \dot{-} y = \begin{cases} x - y & \text{se } x \geq y \\ 0 & \text{altrimenti} \end{cases}$

$x \dot{-} 0 = x$

$x \dot{-} (y+1) = pred(x \dot{-} y)$

Oss $\pi \in S^n$ $f : \mathbb{N}^n \to \mathbb{N}$ p.r.

allora $(x_1 \dots x_n) \mapsto f(x_{\pi(1)} \dots x_{\pi(n)})$

è p.r.

Prof $S \subseteq \mathbb{N}^n$ è un predicato p.r. se esiste $f : \mathbb{N}^n \to \mathbb{N}$ p.r. t.c.

$\bar{x} \in S \iff f(\bar{x}) = 0$

- $\text{zero?}(x) = \begin{cases} 0 & \text{se } x = 0 \\ 1 & \text{se } x \neq 0 \end{cases}$

$f(x_1, \dots x_n, 0) = h(x_1, \dots x_n)$ ← cost. 0

$f(x_1, \dots x_n, y+1) = g(x_1, \dots x_n, y, f(x_1, \dots x_n, y))$ ← cost. 1

Oss. $P \subseteq \mathbb{N}^n$ è un predicato p.r. sse

$$f(\bar{x}) = \begin{cases} 0 & \text{se } \bar{x} \in P \\ 1 & \text{se } \bar{x} \notin P \end{cases} \quad \text{è p.r.}$$

Oss $P \subseteq \mathbb{N}^n$ p.r. allora $\mathbb{N}^n \setminus P$ è p.r.

$$f(\bar{x}) = \begin{cases} 0 & \text{se } \bar{x} \in P \\ 1 & \text{se } \bar{x} \notin P \end{cases} \quad \text{è p.r.}$$

considero $g(\bar{x}) = 1 \dot{-} f(\bar{x})$

Oss Se $P, Q \subseteq \mathbb{N}^n$ p.r. allora

$P \cap Q$ e $P \cup Q$ sono p.r.

Quindi per ogni k i sottoinsiemi p.r. di $\mathbb{N}^n$ sono un'algebra di Boole

- Sono r.a.

$x < y$ $\quad x \le y$ $\quad x = y$ $\quad x \neq y$

$s(x) \dot{-} y$ $\quad x \dot{-} y$ $\quad x \le y \wedge y \le x$ $\quad \neg(x = y)$

- Definizione per casi $\quad h(\bar{x}) = \begin{cases} 0 & \bar{x} \in P \\ 1 & \bar{x} \notin P \end{cases}$

$P \subseteq \mathbb{N}^r$ r.a. $\quad f, g : \mathbb{N}^r \to \mathbb{N}$ a.r.

$$\mathbb{N}^r \longrightarrow \mathbb{N}$$

$$\bar{x} \longmapsto \begin{cases} f(\bar{x}) & \text{se } \bar{x} \in P \\ g(\bar{x}) & \text{altrimenti} \end{cases}$$

$$g(\bar{x}) \cdot h(\bar{x}) + f(\bar{x}) \cdot (1 \dot{-} h(\bar{x}))$$

- $x \bmod y = z \quad \leftrightarrow \quad z < y \wedge y \mid x - z$

$$x \bmod y = \begin{cases} x & \text{se } x < y \\ (x \dot{-} y) \bmod y & \text{altrimenti} \end{cases}$$

$f(x_1, \dots, x_n, 0) = h(x_1, \dots, x_n)$

$f(x_1, \dots, x_n, y+1) = g(x_1, \dots, x_n, y, f(x_1, \dots, x_n, y))$

$$x+1 \bmod y = \begin{cases} \overbrace{(x \bmod y)+1}^{\alpha} & (x \bmod y)+1 < y \\ 0 & \end{cases}$$

$$h(x,y,\alpha) = \begin{cases} \alpha+1 & \alpha+1 < y \\ 0 & \text{else} \end{cases}$$

- $f : \mathbb{N}^{k+1} \to \mathbb{N}$ p.r.

$$\mathbb{N}^{k+1} \longrightarrow \mathbb{N} \qquad (\Pi)$$

$$(\bar{x}, y) \longmapsto \sum_{i<y} f(\bar{x}, i) \qquad i \text{ l.a.}$$

$$\sum_{i<y+1} f(\bar{x}, i) = \left(\sum_{i<y} f(\bar{x}, i) \right) + f(\bar{x}, y)$$

- $P \subseteq \mathbb{N}^{k+1}$

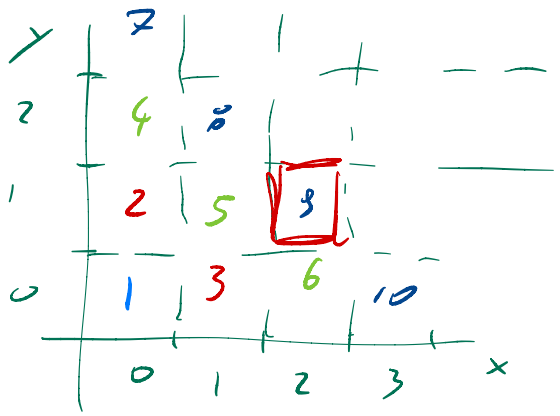
$$\mathbb{N}^{k+1} \longrightarrow \mathbb{N}$$

$$(\bar{x}, y) \longmapsto \#\{ (\bar{x}, i) \mid i<y \wedge (\bar{x}, i) \in P \} \qquad \text{i.l.}$$

- $x \operatorname{div} y = \left\lfloor \frac{x}{y} \right\rfloor$

$$= \#\{ i < x \mid i+1 \bmod y = 0 \}$$

- $\text{cons}(x,y) = \frac{(x+y)(x+y+1)}{2} + x + 1$



- car , cdr

$\text{car}(\text{cons}(x,y)) = x$

$\text{cdr}(\text{cons}(x,y)) = y$

Fatto

$f : \mathbb{N}^n \to \mathbb{N} \cup \{\perp\}$ computabile

allora esiste P predicato l.r.

$f(\bar{x}) = y \iff \exists z \; P(\bar{x}, y, z)$

Sia $z = cons(x,y)$

$$x + y = \#\left\{ n < z \mid \frac{n(n+1)}{2} \leq z \right\} + 1 \quad ?$$

$$x = cons(x+y) \div \frac{(x+y)(x+y+1)}{2} \div 1$$

Def

$$\langle x_1 \dots x_k \rangle = cons(x_1, cons(x_2, \dots cons(x_k, 0)\dots))$$

$$\langle\rangle = 0$$

$$\langle x_1, x_2, \dots x_n \rangle = cons(x_1, \langle x_2 \dots x_n \rangle)$$

Es. Sono p.r.

- lunghezza (l)
- $cat(\langle x_1 \dots x_n \rangle, \langle y_1 \dots y_h \rangle) = \langle x_1 \dots x_n y_1 \dots y_h \rangle$
- $head(l, n) = \langle l_1 \dots l_n \rangle$

$$l_i = \begin{cases} \text{i-esimo el. di } l & \text{se } i \leq lung(l) \\ 0 & \text{altrimenti} \end{cases}$$