$\underline{\text{DaL}}\ (PA,\ Q)$

Q_1 $\quad \forall x\ \neg s(x) = 0$

Q_2 $\quad \forall x\ \forall y\ \ s(x) = s(y) \to x = y$

Q_3

$\vdots$ $\qquad$ def. ric. di $+$ e $\cdot$

Q_6

Q_7 $\quad \forall x\ \ x = 0 \ \vee\ \exists y\ \ x = s(y)$

PA_7 $\quad (\forall x\ \varphi(x) \to \varphi(x+1)) \to \forall y\ \varphi(y)$

$\underline{\text{Def}}$ (φ decidibile in T)

T decide φ se $\quad T \vdash \varphi \quad$ o $\quad T \vdash \neg\varphi$

$\underline{\text{Prop}}$ Sia φ formula $\underline{\text{chiusa}}$ di cl. Δ_0^0 allora φ è *decidibile* in Q. Più precisamente

$$\mathbb{N} \models \varphi \iff Q \vdash \varphi$$

$$\mathbb{N} \models \neg\varphi \iff Q \vdash \neg\varphi$$

Sia φ $\underline{\text{CHIUSA}}$ di classe Σ_1^0, allora

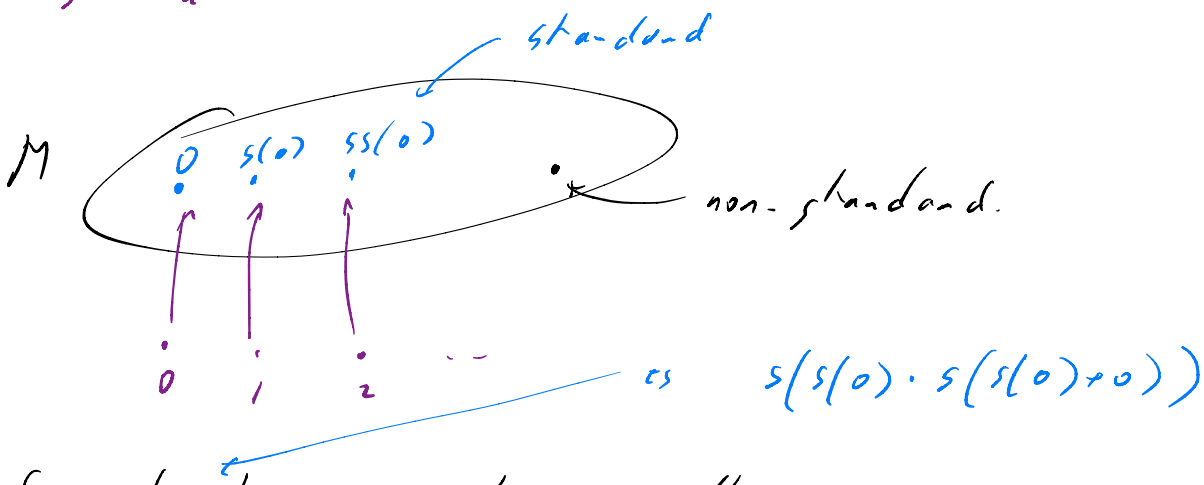
$$\mathbb{N} \models \varphi \iff Q \vdash \varphi$$

Notazione $n \in \mathbb{N}$ $\quad \overline{n} \overset{\text{def.}}{\equiv} \underbrace{s(s(\dots s(0)\dots))}_{n \text{ successori}}$

Def Sia $M \models Q$.

Chiamiamo $x \in M$ **standard** se $M \models x = \overline{n}$ per qualche $n \in \mathbb{N}$

e **non-standard** altrimenti.



Lemma Sia t termine chiuso allora

$Q \vdash t = \overline{n}$ con $n = \{\}^t_{\mathbb{N}}$

Dim per induzione sulla struttura di t, dove verifichiamo che dati $m, n \in \mathbb{N}$

$Q \vdash s(\overline{n}) = \overline{s(n)}$ ✓

$Q \vdash \overline{m} + \overline{n} = \overline{m+n}$

$Q \vdash \overline{m} \cdot \overline{n} = \overline{m \cdot n}$

} es. (ind. su n)

□

$t_1 = t_2$

Lem φ atomica chiusa $\Rightarrow$ φ è decidibile in Q

$$\mathbb{N} \models \varphi \iff Q \vdash \varphi$$
$$\mathbb{N} \models \neg\varphi \iff Q \vdash \neg\varphi$$

Dim Sia $n_1 = \xi_{\mathbb{N}}\, t_1$ $\qquad$ $n_2 = \xi_{\mathbb{N}}\, t_2$ allora

$$\mathbb{N} \models \varphi \iff n_1 = n_2$$

- se $n_1 = n_2$ allora $\quad Q \vdash t_1 = \overline{n_1}$
$$t_2 = \overline{n_2}$$
$$Q \vdash t_1 = t_2$$

- se $n_1 \neq n_2$ basta dim. che $\quad Q \vdash \neg\, \overline{n_1} = \overline{n_2}$

senza perdita di gen. $n_1 < n_2$

induzione su n_2

- caso $n_2 = 0$ ✓

- caso $n_2 = m_2 + 1$

$n_1 = 0$

$Q \vdash \neg\, 0 = \overline{m_2 + 1}$

$\neg\, 0 = S(\overline{m_2})$ ✓

$n_1 \neq 0$
$\|$
$m_1 + 1$

$Q \vdash \neg\, \overline{m_1 + 1} = \overline{m_2 + 1}$

$Q \vdash \neg\, S(\overline{m_1}) = S(\overline{m_2})$

⋮

$Q \vdash \neg \bar{n}_1 = \bar{n}_2$

ipotesi induttiva.

Def $t_1 \leq t_2 \overset{def.}{\equiv} \exists x \; x + t_1 = t_2$

Cosa 1 ✗ $Q \vdash \forall x \forall y \; x \leq s(y) \to (x = s(y) \vee x \leq y)$

Cosa 2 ✓ $M \models Q$, $x \in M$ non-standard

$M \models \bar{n} \leq x$

Lem

$\forall n \in \mathbb{N} \quad Q \vdash \forall a \left(a \leq \bar{n} \leftrightarrow (a = 0 \vee a = s(0) \vee \dots \vee a = \bar{n}) \right)$

Dim Per ind. su n.

- $n = 0$ $\quad Q \vdash a \leq 0 \leftrightarrow a = 0$

$\exists x \; x + a = 0$

$\exists x \; x + a = 0$ $\qquad a \neq 0$

$\downarrow$ $\qquad \downarrow$

$\exists x \; x + s(s) = 0$ $\leftarrow$ $a = s(s)$ $\exists$

$s(x + s) = 0$ ⇝

- $n = m + 1$

$Q \vdash a \leq \overline{m+1} \leftrightarrow a = 0 \vee (\exists b \; a = s(b) \wedge b \leq \bar{m})$

$Q \vdash a \leq \overline{m+1} \leftrightarrow a = 0 \vee \exists b \; a = s(b) \wedge (b = 0 \vee \dots \vee b = \bar{m})$

$(\exists b \; a = s(b) \wedge b = 0) \vee (\exists b \; a = s(b) \wedge b = s(0)) \vee \dots$

$a = s(0)$ $\qquad$ $a = s(s(0))$ $\qquad \dots$

$Q \vdash a \leq \overline{m+1} \leftrightarrow a = 0 \vee (\exists b \; a = s(b) \wedge b \leq \bar{m})$

$\rightarrow \exists x \; x + a = \overline{m+1}$ $\qquad$ $a \neq 0 \rightarrow a = s(b)$

TESI: $\exists y \; y + b = \bar{m}$ $\qquad$ $\exists x \; x + s(b) = \overline{m+1}$

$s(x + b) = \overline{m+1}$

$x + b = \bar{m}$

$\rightarrow$ esercizio.

∎

Lemma $M \models Q$ $a \in M$ non-standard $n \in \mathbb{N}$

allora $M \models \bar{n} \leq a$

Dim $M \models \exists x \; x + \bar{n} = a$

- se $n = 0$ $M \models \exists x \; x + 0 = a$
- se $n = m+1$ $M \models \exists x \; \underbrace{x + \overline{m+1}}_{s(x+\bar{m})} = a$

$s(x + \bar{m}) = s(b)$

$x + \bar{m} = b$

□

Prop Sia φ formula chiusa di cl. Δ_0 allora φ è *decidibile* in Q. Cioè, più precisamente:

$$\mathbb{N} \models \varphi \iff Q \vdash \varphi$$
$$\mathbb{N} \models \neg\varphi \iff Q \vdash \neg\varphi$$

Dim $\Leftarrow$ immediata.

$\Rightarrow$ Induzione sulla struttura di φ.

- $\varphi = \psi_1 \wedge \psi_2$

$\mathbb{N} \models \varphi \Rightarrow \mathbb{N} \models \psi_1 \wedge \mathbb{N} \models \psi_2$

$Q \vdash \psi_1$ $Q \vdash \psi_2 \Rightarrow Q \vdash \psi_1 \wedge \psi_2$

$\mathbb{N} \models \neg\varphi \Rightarrow \mathbb{N} \models \neg\psi_1 \vee \mathbb{N} \models \psi_2$

$\Updownarrow \qquad \Updownarrow$

$Q \vdash \neg\psi_1 \vee Q \vdash \neg\psi_2$

$Q \vdash \neg(\psi_1 \wedge \psi_2) \qquad Q \vdash \neg(\psi_1 \wedge \psi_2)$

$\vdots$

$\{t\}_{\mathbb{N}} = n \qquad Q \vdash t = \bar{n}$

- $\varphi = \forall x \leq t \; \psi$

$x \leq \bar{n}$

$Q \vdash \varphi \leftrightarrow \forall x \; (x = 0 \vee x = 1 \vee \dots \vee x = \bar{n}) \rightarrow \psi$

$Q \vdash \varphi \leftrightarrow \forall x \; (x = 0 \rightarrow \psi) \wedge (x = 1 \rightarrow \psi) \wedge \dots$

$Q \vdash \varphi \leftarrow (\forall x \; x = 0 \rightarrow \psi) \wedge (\forall x \; x = 1 \rightarrow \psi) \wedge \dots$

$\psi[0/x] \qquad \psi[1/x]$

$M \models \forall x \; x = 0 \rightarrow \psi$

$\forall a \in M \quad M \models \{v[a/x]\}(x = 0 \rightarrow \psi)$

$\Downarrow$

$M \models \{v[0/x]\}(x = 0 \rightarrow \psi)$

$M \models \{v\}(x = 0 \rightarrow \psi)[0/x]$

$M \models \{a\}(0 = 0 \rightarrow \psi[0/x])$

$M \models \{v\}\,\psi[0/x]$

- $\varphi = \exists x \leq t^{=n}\ \psi$

$Q \vdash \varphi \leftrightarrow \exists x\,(x = 0 \vee x = 1 \vee \dots \vee x = \overline{n}) \wedge \psi$

$Q \vdash \varphi \leftrightarrow \exists x\,(x = 0 \wedge \psi) \vee (x = 1 \wedge \psi) \vee \dots$

$Q \vdash \varphi \leftrightarrow \underbrace{(\exists x\ x = 0 \wedge \psi)}_{\psi[0/x]} \vee \underbrace{(\exists x\ x = 1 \wedge \psi)}_{\psi[1/x]} \vee \dots$

□