Prop

Sia φ <u>chiusa</u> di classe Σ^0_1, allora

$$\mathbb{N} \models \varphi \iff Q \vdash \varphi$$

Dim $\varphi \equiv \exists x_1 \dots \exists x_k \, \psi$ Δ^0_0

- $\mathbb{N} \models \varphi \Rightarrow$ esistono $n_1 \dots n_k$ t.c.

$$\mathbb{N} \models \psi[\bar{n}_1/x_1, \dots \bar{n}_k/x_k]$$

$$Q \vdash \psi[\bar{n}_1/x_1, \dots \bar{n}_k/x_k]$$

$$Q \vdash \exists x_1 \dots \exists x_k \, \psi$$

- $Q \vdash \varphi \Rightarrow \mathbb{N} \models \varphi$ ∎

Lem Sia $f : \mathbb{N}^k \to \mathbb{N}$ ricorsiva TOTALE

allora esiste φ di classe Σ^0_1 t.c.

$$\forall x_1 \dots x_k \quad Q \vdash \forall y \; \varphi(\bar{x}_1 \dots \bar{x}_k, y) \leftrightarrow y = \overline{f(x_1 \dots x_k)}$$

$$\downarrow$$

$$\exists! y \; \varphi(\bar{x}_1 \dots \bar{x}_k, y)$$

$$\forall x_1 \dots x_k \quad Q \vdash \forall y \; \varphi(\bar{x}_1 \dots \bar{x}_k, y) \leftrightarrow y = \overline{f(x_1 \dots x_k)}$$

$$\forall x_1 \dots x_k \; \forall y \; \left(Q \vdash \varphi(\bar{x}_1 \dots \bar{x}_k, \bar{y}) \right) \leftrightarrow y = f(x_1 \dots x_k)$$

$\exists z_1 \dots z_h \; \theta$ $\quad \theta \in \Delta_0$

$$\varphi \equiv \exists N \Big(\exists z_1 \le N \dots \exists z_h \le N \; \theta(z_1 \dots z_h, x_1 \dots x_k, y)$$

$$\wedge \forall z_1 \le N \dots \forall z_h \le N \; \forall t$$

$$\theta(z_1 \dots z_h, x_1 \dots x_k, t) \to t = y \Big)$$

Fisso $x_1 \dots x_k \in \mathbb{N}$, fisso $M \models Q$ t.c.

$$M \models \forall y \; \varphi(\bar{x}_1 \dots \bar{x}_k, y) \leftrightarrow y = \overline{f(x_1 \dots x_k)}$$

$$\exists y \in M \quad M \models \neg \left(\varphi(\bar{x}_1 \dots \bar{x}_k, y) \leftrightarrow y = \overline{f(x_1 \dots x_k)} \right)$$

Lemma Sea $M \neq Q$, $a \in M$ no-st., $n \in \mathbb{N}$

entonces $M \models \bar{n} \leq a$

Dem $M \models \bar{n} \leq a$

$M \models \exists x \; x + \bar{n} = a$

Inducción en n

- $n = 0$ inmediato

- $n = m + 1$ $M \models \exists x \; x + \bar{m} = a$

$M \models \exists x \; \underbrace{x + \overline{m+1}}_{x + S(\bar{m})} = a = S(m)$

$\parallel$

$S(x + \bar{m})$

□

Primo teorema di incompletezza di Gödel

Teo PA è incompleta.

Numerazione di Gödel

L'alfabeto della formula è

$\neg$	$\rightarrow$	$\forall$	$=$	0	S	$+$	$\cdot$	x_i
$\langle 0,0\rangle$	$\langle 1,0\rangle$	$\langle 2,0\rangle$	-	-	-			$\langle 8,i\rangle$
$\#\neg$	$\#\rightarrow$							$\#x_i$

- N. di Gödel dei termini

- t. atomici:

$$\ulcorner 0 \urcorner = \langle \#0 \rangle$$

$$\ulcorner x_i \urcorner = \langle \# x_i \rangle$$

- t. composti:

$$\ulcorner S(t) \urcorner = \langle \# S, \ulcorner t \urcorner \rangle$$

$$\ulcorner t_1 + t_2 \urcorner = \langle \# +, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$$

- formule

$$\ulcorner t_1 = t_2 \urcorner = \langle \# =, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$$

$$\ulcorner \varphi \rightarrow \psi \urcorner = \langle \# \rightarrow, \ulcorner \varphi \urcorner, \ulcorner \psi \urcorner \rangle$$

$\forall x_i \varphi = \langle \#\forall, \#x_i, \ulcorner\varphi\urcorner \rangle$

Lem

Il predicato $Fon(n) \equiv \exists \varphi \; n = \ulcorner\varphi\urcorner$ è p.r.

Dim

$Fon(n) \leftrightarrow$ n è una lista $\longrightarrow n \neq 0$

$can(n) \in$ alfabeto

e della forma

- 0 $can(n) = \#\neg \wedge lungh(n) = 2$
  $\wedge \; Fon(nth(1, n))$
- 0 ...
- ⋮

$Ter(n) \leftrightarrow$ ...

Lem $Num : \mathbb{N} \longrightarrow \mathbb{N}$

$n \longmapsto \ulcorner\overline{n}\urcorner$ è p.r.

Dim

$Num(0) = \ulcorner 0 \urcorner$

$Num(n+1) = \langle \# S, Num(n) \rangle$

Lm $\quad Sub : \mathbb{N}^3 \longrightarrow \mathbb{N}$

$$(\ulcorner\varphi\urcorner, i, \ulcorner t\urcorner) \longmapsto \ulcorner \varphi(t/x_i)\urcorner$$

Dim

$Sub(n,i,m) =$

Se $\neg(Fun(n) \times Term(m))$ allora 0

Se $Term(n)$

:

- se $can(n) = \# + $ allora

!

$$\langle \#+, Sub(nth(1,n), i, m), Sub(nth(2,n), i, m)\rangle$$

- se $can(n) = \ulcorner x_k \urcorner$ allora

se $k = i$ $\quad m$

se $k \neq i$ $\quad \ulcorner x_k \urcorner$

Se $Fun(n)$

:

Lemma (di diagonalizzazione)

Data una f. aritmetica φ esiste ψ t.c.

$$Q \vdash \psi \leftrightarrow \varphi[\overline{\ulcorner\psi\urcorner}/x_1]$$

$$\varphi(\overline{\ulcorner\psi\urcorner})$$

Dim

Idea $\pi(x) = \varphi(x(x))$

$\pi(\pi) = \varphi(\pi(\pi))$

$\pi : \ulcorner\alpha\urcorner \longmapsto \ulcorner\alpha(\overline{\ulcorner\alpha\urcorner}/x_1)\urcorner$

$x \longmapsto Sub(x, 1, Num(x))$

Esiste θ di classe Σ^0_1 t.c.

$\forall n \quad Q \vdash \theta[\overline{n}/x_1] \leftrightarrow x_2 = \overline{\pi(n)}$